First principles justification of a "single wave model" for electrostatic instabilities

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The nonlinear evolution of a unstable electrostatic wave is considered for a multispecies Vlasov plasma. From the singularity structure of the associated amplitude expansions, the asymptotic features of the electric field and distribution functions are studied in the limit of weak instability, i.e., $\gamma \rightarrow 0^+$ where γ is the linear growth rate. The asymptotic electric field is monochromatic at the wavelength of the linear mode with a nonlinear time dependence. The structure of the distributions outside the resonant region is given by the linear eigenfunction but in the resonant region the distribution is nonlinear. The details depend on whether the ions are fixed or mobile; in either case this generally derived physical picture corresponds to the single wave model originally proposed by O'Neil, Winfrey, and Malmberg [Phys. Fluids **14**, 1204 (1971)] for the special case of a cold weak beam instability in a plasma of fixed ions.
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I. INTRODUCTION

Recently, we have studied the collisionless nonlinear evolution of a weakly unstable mode.[1–3] In this setting, one expects that the small growth rate limit ($\gamma \rightarrow 0$) will determine a characteristic scaling exponent for the mode electric field $E \sim \gamma^{\beta}$ and the theoretical determination of β is an interesting aspect of the problem. This exponent has a somewhat controversial history in the plasma literature (see Ref. 1 for additional discussion) from which two influential contributions provided our initial motivation. For an infinite one-dimensional system with fixed ions, O'Neil, Winfrey, and Malmberg (OWM) considered the instability due to a weak cold electron beam and formulated a simplified dynamical model—the single wave model.[4] In their picture, the unstable mode saturates by trapping the beam particles and the resulting electric field exhibits the "trapping scaling," $E \sim \gamma^2$ (γ is the initial linear growth rate of the mode). Subsequently, Simon and Rosenbluth studied the corresponding problem for a bump-on-tail beam instability in a finite one-dimensional system also with fixed ions.[5] They sought to determine the time-asymptotic state from a multiple time scales expansion, and predicted a qualitatively different scaling behavior $E \sim \gamma^{1/2}$. Their calculation is marked by technical difficulties due to singular coefficients that appear in the perturbation theory, necessitating a variety of recipes to obtain finite results.

We have developed a systematic approach to the $\gamma \rightarrow 0$ regime of such problems that can treat single mode instabilities in a uniform fashion; the singularities encountered by Simon and Rosenbluth are given a direct physical interpretation and also handled in a mathematically natural manner. This research has been further motivated by the recognition that the difficulties encountered by Simon and Rosenbluth are generic to a wide class of problems in which a weakly unstable mode evolves while coupled to some neutrally stable modes.[6–8] (In Vlasov theory, the neutral modes are supplied by the van Kampen continuous spectrum.) In all of these problems, in the $\gamma \rightarrow 0$ limit, the strong nonlinear interaction with the neutral modes produces a kind of singular perturbation problem whose structure determines the scaling exponent β as well as many other features of the asymptotic solutions.

We use series expansions to represent the nonlinear solutions, and apply the series to analyze the dynamics on the unstable manifold of problem. This amounts to considering initial conditions in which only the unstable mode is excited and leads to the simplest setting for treating the dynamics because the unstable manifold is two-dimensional. However, because the manifold is also global and invariant, we are not in principle restricting ourselves to short-time scale information. The full series are viewed as representing exact nonlinear solutions; we do not necessarily expect that they can be truncated at some low order to obtain accurate approximate solutions. Rather it is important to treat them *whole* and this emphasizes the need to only draw conclusions that can be shown to hold to all orders (if possible).

This approach has now been successfully worked out for electrostatic instabilities in a Vlasov plasma with several mobile species; generalizing an earlier investigation that assumed fixed ions.[1] In our analysis, there are two interrelated expansions that need to be studied: The expansion of the equation describing the evolution of the wave amplitude $A(t)$

$$\dot{A} = \lambda A + p_1 A|A|^2 + p_2 A|A|^4 + \cdots , \tag{1}$$

and the expansion for the particle distribution functions F

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$$F(x,v,t)=F_0(v)+A(t)e^{ikx}\psi(v)+A^*(t)e^{-ikx}\psi^*(v)+|A|^2h_{0,0}(v)+A^2e^{i2kx}h_{2,0}(v)+(A^*)^2e^{-i2kx}h^*_{2,0}(v)+\cdots. \quad (2)$$

In each of these expansions, the $\gamma\to 0$ limit generates various coefficient singularities which we are able to handle mathematically in a simple uniform way that works to all orders. In a previous paper, this program was carried out for the first expansion (1) in a quite detailed analysis with a very simple conclusion: Set $|A(t)|=\gamma^\beta r(\gamma t)$ and rewrite the expansion (1) in terms of r.[3] The factors of γ^β thus introduced exactly remove all singularities in the expansion coefficients to all orders provided one chooses $\beta=\frac{5}{2}$ (there are special exceptions, such as the limit of fixed ions where the correct exponent is $\beta=2$). The induction proof that there is actually a value of β that achieves this is the main result of Ref. 3. The *physical* significance of the singularities in the coefficients $\{p_j\}$ is to set this scaling exponent for the mode amplitude, and this determines the overall scaling of the electric field through the Poisson equation. Thus this result predicts the electric field scaling $E\sim\gamma^{5/2}$ universally for these instabilities (including beam-plasma and two-stream configurations) throughout the dynamical evolution and in the time-asymptotic state. In special cases, such as infinitely massive ions (fixed-ions), the electric field gives back the familiar ''trapping scaling'' $E\sim\gamma^2$ as expected. The shift from $\beta=2$ to $\beta=\frac{5}{2}$ as the true asymptotic scaling due to the finite mass ions is a significant conclusion regarding the physical character of the wave. Since previous studies of the asymptotic scaling of single wave instabilities have focused primarily on infinitely massive ions, this shift was not anticipated and was discovered purely as a consequence of the singularities in the amplitude equation, Eq. (1).

We build on this result with two main goals. The first goal is to demonstrate that the same choices for β also serve to cure the singularities exhibited in the second expansion for F. This is an important conclusion and fortunately it is comparatively easy to demonstrate given various estimates on the coefficients $\{h_{m,j}(v)\}$ developed in our previous paper. The expansions for F only yield singularities for velocities in the so-called resonant region near the linear phase velocity and in this local neighborhood the expansion for F can be rewritten in terms of a ''fast'' velocity scale variable u; this extracts from the each expansion coefficient $h_{m,j}(v)$ the singularity it will exhibit as $\gamma\to 0$. These singularities are then balanced against the factors of γ^β coming from the rescaling of A and one sees that the factors of γ^β are sufficient to remove all the velocity space singularities in the resonance region. In this way, the $\gamma\to 0$ limit is sensibly handled in both expansions and to all orders. It is worth noting that in other problems with weakly unstable modes coupled to neutrally stable modes the singularities in the amplitude expansion have been successfully understood in the same way.[8]

A second goal of the present paper is to discuss the physical characteristics of the asymptotic solutions once the limit $\gamma\to 0$ has been mathematically understood. We examine the resulting asymptotic expressions for F (in both resonant and nonresonant regions of velocity) and for the electric field E and note something simple and interesting. Even though we work in a more general physical setting than OWM (and, therefore, are analyzing a much larger class of instabilities), the *derived* asymptotic forms for E and F exhibit rather precisely the features that OWM *postulate* as the basis of their single wave model, i.e.,

(1) The electric field is monochromatic (one k value).
(2) The *nonresonant* particles respond to this electric field in an essentially linear fashion, i.e., the x and v dependence in given by the linear eigenfunction.
(3) The *resonant* particle distribution, expressed in the velocity variable u, has a much more complicated structure that is clearly determined by nonlinear processes, e.g., particle trapping. One interesting feature of our resonant particles is that the wave number content of the distribution is vastly simpler when the ions are mobile compared to the fixed ion limit.

A significant aspect of this conclusion is that these features have been derived by a systematic analysis carried out to all orders.

Assuming these features allowed OWM to formulate their single wave model (SWM) for a cold beam instability which was a much simpler system of equations for the problem that nevertheless should preserve the essential physical features of the problem. The SWM is not analytically solvable but it is vastly simpler to treat numerically and has been a useful model for many theoretical studies since its introduction.[9–16] The same features have now been derived for a much larger class of weakly unstable modes and this provides a ''first principles'' justification for constructing and studying analogous SWM type models for electrostatic instabilities quite generally.

In the remainder of this introduction we review our notation and in Sec. II we summarize the needed conclusions of Ref. 3 regarding the singularities of the expansions. Section III applies these conclusions to the distributions and electric field, and Sec. IV contains a final discussion. As this paper was being completed, we learned of the interesting recent work by del-Castillo-Negrete who has given an different derivation of the single wave picture using matched asymptotic methods to treat the resonant and nonresonant particles.[17] As in the original work of O'Neil *et al.*, del-Castillo-Negrete allows only mobile electrons, and moreover, restricts attention to instabilities associated with so-called ''inflection point modes.''[18,19]

A. Notation

Our notation follows Ref. 3; we consider a one-dimensional, multispecies Vlasov plasma defined by

$$\frac{\partial F^{(s)}}{\partial t}+v\frac{\partial F^{(s)}}{\partial x}+\kappa^{(s)}E\frac{\partial F^{(s)}}{\partial v}=0, \quad (3)$$

$$\frac{\partial E}{\partial x}=\sum_s\int_{-\infty}^{\infty}dv\,F^{(s)}(x,v,t). \quad (4)$$

Here x, t, and v are measured in units of u/ω_e, ω_e^{-1} and u, respectively, where u is a chosen velocity scale and ω_e^2

$=4\pi e^2 n_e/m_e$. The plasma length is L with periodic boundary conditions and we adopt the normalization

$$\int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv F^{(s)}(x,v,t) = \left(\frac{z_s n_s}{n_e}\right) L, \tag{5}$$

where $q_s = e z_s$ is the charge of species s and $\kappa^{(s)} \equiv q_s m_e / e m_s$. Note that $\kappa^{(e)} = -1$ for electrons and that the normalization (5) for negative species makes the distribution function negative.

Let $F_0(v)$ and $f(x,v,t)$ denote the multicomponent fields for the equilibrium and perturbation, respectively, and κ the matrix of mass ratios

$$f \equiv \begin{pmatrix} f^{(s_1)} \\ f^{(s_2)} \\ \vdots \end{pmatrix}, \quad F_0 \equiv \begin{pmatrix} F_0^{(s_1)} \\ F_0^{(s_2)} \\ \vdots \end{pmatrix},$$

$$\kappa \equiv \begin{pmatrix} \kappa^{(s_1)} & 0 & 0 & \cdots \\ 0 & \kappa^{(s_2)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{6}$$

then systems (3) and (4) can concisely be expressed as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + \mathcal{N}(f), \tag{7}$$

where the linear operator is defined by

$$\mathcal{L} f = \sum_{l=-\infty}^{\infty} e^{ilx} (L_l f_l)(v), \tag{8}$$

$$(L_l f_l)(v) = \begin{cases} 0 & l=0 \\ -il\left[v f_l(v) + \kappa \cdot \eta_l(v) \sum_s \int_{-\infty}^{\infty} dv' f_l^{(s)}(v') \right] & l \neq 0, \end{cases} \tag{9}$$

with $\eta_l(v) \equiv -\partial_v F_0 / l^2$, and the nonlinear operator $\mathcal{N}$ is

$$\mathcal{N}(f) = \sum_{m=-\infty}^{\infty} e^{imx} \sum_{l=-\infty}^{\infty}{}' \frac{i}{l} \left(\kappa \cdot \frac{\partial f_{m-l}}{\partial v} \right) \times \sum_{s'} \int_{-\infty}^{\infty} dv' f_l^{(s')}(v'). \tag{10}$$

In the spatial Fourier expansion (8), l denotes an integer multiple of the basic wave number $2\pi/L$, and the primed summation in Eq. (10) omits the $l=0$ term. The notation $\kappa \cdot \eta_l(v)$ or $\kappa \cdot \partial_v f_{m-l}$ denotes matrix multiplication. For two multicomponent fields of (x,v), e.g., $B = (B^{(s_1)}, B^{(s_2)}, B^{(s_3)}, \ldots)$ and $D = (D^{(s_1)}, D^{(s_2)}, D^{(s_3)}, \ldots)$, we define an inner product by

$$(B,D) \equiv \sum_s \int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv B^{(s)}(x,v)^* D^{(s)}(x,v) = \int_{-L/2}^{L/2} dx \langle B,D \rangle, \tag{11}$$

where

$$\langle B,D \rangle \equiv \sum_s \int_{-\infty}^{\infty} dv B^{(s)}(x,v)^* D^{(s)}(x,v). \tag{12}$$

The spectral theory for $\mathcal{L}$ is well established and the facts needed for our analysis are easily summarized. The eigenvalues $\lambda = -ilz$ of $\mathcal{L}$ are determined by the roots $\Lambda_l(z) = 0$ of the ''spectral function''

$$\Lambda_l(z) \equiv 1 + \int_{-\infty}^{\infty} dv \frac{\Sigma_s \kappa^{(s)} \eta_l^{(s)}(v)}{v - z}. \tag{13}$$

If the contour in Eq. (13) is replaced by the Landau contour for Im$(z)<0$ then we have the linear dielectric $\epsilon_l(z)$; for Im$(z)>0$, $\Lambda_l(z)$, and $\epsilon_l(z)$ are the same function. The eigenvalues can be either real or complex depending on the symmetry and shape of the equilibrium.

Associated with an eigenvalue $\lambda = -ilz$ is the multicomponent eigenfunction $\Phi(x,v) = e^{ilx}\psi(v)$ where

$$\psi(v) = -\frac{\kappa \cdot \eta_l}{v - z}. \tag{14}$$

There is also an associated adjoint eigenfunction $\tilde{\Psi}(x,v) = e^{ilx}\tilde{\psi}(v)/L$ satisfying $(\tilde{\Psi}, \Psi) = 1$ with

$$\tilde{\psi}(v) = -\frac{1}{\Lambda_l'(z)^*(v - z^*)}. \tag{15}$$

Note that all components of $\tilde{\psi}(v)$ are the same. The normalization in Eq. (15) assumes that the root of $\Lambda_l(z)$ is simple and is chosen so that $\langle \tilde{\psi}, \psi \rangle = 1$. The adjoint determines the projection of $f(x,v,t)$ onto the eigenvector, and this projection defines the time-dependent amplitude of Ψ, i.e., $A(t) \equiv (\tilde{\Psi}, f)$.

II. PREVIOUS RESULTS

In this section, we provide a brief synopsis of the approach and most relevant conclusions from our previous paper.[3] The equilibrium $F_0(v)$ is assumed to support a ''single'' unstable mode. With translation symmetry and periodic boundary conditions, this is the simplest instability problem that can be posed. An unstable mode exists if the spectral function $\Lambda_k(z)$ has a root $z_0 \equiv i\lambda/k = v_p + i\gamma/k$ in the upper half of the complex plane ($\gamma > 0$). Henceforth, let k denote the wave number of this unstable mode that is associated with the root $\Lambda_k(z_0) = 0$ which we assume to be simple, i.e., $\Lambda_k'(z_0) \neq 0$. The corresponding eigenvector is

$$\Psi(x,v) = e^{ikx}\psi(v) = e^{ikx}\left(-\frac{\kappa \cdot \eta_k}{v - z_0} \right). \tag{16}$$

The root $z_0 = v_p + i\gamma/k$ determines the phase velocity $v_p = \omega/k$ and the growth rate γ of the linear mode as the real and imaginary parts of the eigenvalue $\lambda = -ikz_0 = \gamma - i\omega$.

If λ is an unstable eigenvalue $[\mathrm{Re}(\lambda) \equiv \gamma > 0]$ with a eigenfunction Ψ, then λ^* is also an unstable eigenvalue and the corresponding eigenfunction is Ψ^*. This corresponds to a wave with wave number $-k$. So the unstable subspace is two-dimensional.

Solutions on the two-dimensional unstable manifold have the form

$$f^u(x,v,t)=[A(t)\psi(v)e^{ikx}+A^*(t)\psi^*(v)e^{-ikx}]+H(x,v,A(t),A^*(t)), \tag{17}$$

where $A(t)\equiv(\tilde{\Psi},f^u)$ evolves according to the amplitude equation

$$\dot{A}=\lambda A+(\tilde{\Psi},\mathcal{N}(f^u)), \tag{18}$$

and self-consistency requires H to satisfy

$$\frac{\partial H}{\partial A}\dot{A}+\frac{\partial H}{\partial A^*}\dot{A}^*=\mathcal{L}H+\mathcal{N}(f^u)-[(\tilde{\Psi},\mathcal{N}(f^u))\Psi+cc], \tag{19}$$

subject to the geometric constraints

$$0=H(x,v,0,0)=\frac{\partial H}{\partial A}(x,v,0,0)=\frac{\partial H}{\partial A^*}(x,v,0,0). \tag{20}$$

The translation symmetry of the model (7) provides important constraints on both the amplitude equation and the form of H.[3] For the amplitude Eq. (18), the right hand side must have the form

$$\lambda A+(\tilde{\Psi},\mathcal{N}(f^u))=Ap(\sigma), \tag{21}$$

where $\sigma\equiv|A|^2$ and $p(\sigma)$ is an unknown function to be determined from the model. Similarly, translational symmetry requires the spatial Fourier components of H to have a special form

$$\begin{aligned}&H_0(v,A,A^*)=\sigma h_0(v,\sigma),\\&H_k(v,A,A^*)=A\sigma h_1(v,\sigma),\\&H_{mk}(v,A,A^*)=A^m h_m(v,\sigma)\quad\text{for } m\geqslant 2,\end{aligned} \tag{22}$$

where $H_{-l}=H_l^*$. These results focus our analysis on a set of functions, $\{p(\sigma),h_m(v,\sigma)\}$, which must be determined from the Vlasov equation.

A. Expansions and singularities

We study $p(\sigma)$ and $\{h_m(v,\sigma)\}$ via the expansions

$$p(\sigma)=\sum_{j=1}^{\infty}p_j\sigma^j,\quad h_m(v,\sigma)=\sum_{j=1}^{\infty}h_{m,j}(v)\sigma^j. \tag{23}$$

The coefficients p_j and $h_{m,j}$ are determined by inserting the expansions into Eqs. (19) and (21) and solving at each order of σ. The resulting recursion relations are given in Ref. 3 and are not required for the present discussion.

The key point is that for both the amplitude equation and the distribution function the expansion coefficients develop singularities in the limit $\gamma\to 0^+$. This can be seen explicitly by reviewing the calculation of the cubic coefficient p_1. From Ref. 3, p_1 depends on $h_{0,0}$ and $h_{2,0}$

$$p_1=-\frac{i}{k}\left[\langle\partial_v\tilde{\psi},\kappa\cdot(h_{0,0}-h_{2,0})\rangle+\frac{\Gamma_{2,0}}{2}\langle\partial_v\tilde{\psi},\kappa\cdot\psi^*\rangle\right], \tag{24}$$

where $\Gamma_{2,0}=\int dv\,h_{2,0}$. The recursion relations determine $h_{0,0}$ and $h_{2,0}$

$$h_{0,0}(v)=-\frac{1}{k^2}\frac{\partial}{\partial v}\left[\frac{\kappa^2\cdot\eta_k}{(v-z_0)(v-z_0^*)}\right], \tag{25}$$

$$h_{2,0}(v)=\frac{1}{2k^2}\left(\frac{\kappa\cdot\partial_v\psi}{v-z_0}\right)+\frac{1}{6k^2}\left(\frac{\kappa\cdot\eta_k}{v-z_0}\right)\left(\frac{\kappa\cdot\eta_k}{v-z_0}\right), \tag{26}$$

and one notes that for $\gamma>0$ these are smooth functions but there are complex poles at z_0 and z_0^* that approach the real axis at $v=v_p$ in the limit $\gamma\to 0^+$. For $h_{2,0}$ all poles lie above the real axis, but $h_{0,0}$ contains poles above and below the axis and this forces the integral $\langle\partial_v\tilde{\psi},\kappa\cdot h_{0,0}\rangle$ in p_1 to diverge as $\gamma\to 0^+$ because of a pinching singularity. For similar reasons, the integral $\langle\partial_v\tilde{\psi},\kappa\cdot\psi^*\rangle$ also diverges but the remaining integrals in p_1 are nonsingular.

A detailed evaluation of this asymptotic structure in p_1 yields the form

$$p_1=\frac{1}{\gamma^4}[c_1(\gamma)-\gamma d_1(\gamma)+\mathcal{O}(\gamma^2)], \tag{27}$$

and c_1 and d_1 are nonsingular functions of γ defined by

$$c_1(\gamma)=-\frac{k}{4\Lambda_k'(z_0)}{\sum_s}'\,\kappa^{(s)}(1-\kappa^{(s)^2})\,\mathrm{Im}\left(\int_{-\infty}^{\infty}dv\,\frac{\eta_k^{(s)}}{v-z_0}\right), \tag{28}$$

$$d_1(\gamma)=\frac{1}{4}-\frac{1}{4\Lambda_k'(z_0)}{\sum_s}'\,\kappa^{(s)}(1-\kappa^{(s)^2})\int_{-\infty}^{\infty}dv\,\frac{\eta_k^{(s)}}{(v-z_0)^2}, \tag{29}$$

where the primed species sum omits the electrons. At $\gamma=0$, c_1 has the limit

$$c_1(0)=-\frac{\pi k}{4\Lambda_k'(z_0)}{\sum_s}'\,\kappa^{(s)}(1-\kappa^{(s)^2})\,\eta_k^{(s)}(v_p), \tag{30}$$

which is typically nonzero yielding a γ^{-4} singularity for p_1. There are at least two special cases of interest for which $c_1(0)=0$; namely, infinitely massive fixed ions ($\kappa^{(s)}=0$ for all $s\neq e$) and flat ion distributions at the resonant velocity ($\eta_k^{(s)}(v_p)=0$ for all $s\neq e$). In such cases, the divergence of p_1 drops to γ^{-3}.

Analogous singularities appear also in the higher order coefficients and grow more severe although their character remains the same. The higher coefficients $h_{m,j}$ exhibit more and more poles, which approach the linear phase velocity as $\gamma\to 0^+$, and these poles generate stronger pinching singularities in the higher coefficients p_j. An important property of the poles in $h_{m,j}$ is that they always have the general form $(v-\alpha)^{-n}$ or $(v-\alpha^*)^{-n}$ with

$$\alpha=z_0+i\gamma\zeta/k=v_p+i\gamma(\zeta+1)/k, \tag{31}$$

where $\zeta>0$ is a purely numerical factor, i.e., the poles always lie along the vertical line $\mathrm{Re}(v)=v_p$.

The explicit calculation of higher order coefficients from recursion relations rapidly becomes prohibitively laborious; however, useful bounds on the singularity of the higher order coefficients are obtained Ref. 3 using an induction argument. More precisely, we find for the amplitude equation

$$\lim_{\gamma\to 0^+} \gamma^{\nu}|p_j|<\infty, \tag{32}$$

for $j\geq 1$ where $\nu=5j-1$ in the generic case with $c_1(0)\neq 0$, and $\nu=4j-1$ in the two special cases mentioned above, fixed ions or flat ion distributions, with $c_1(0)=0$. For the coefficients of the distribution function, the induction argument proves, for $m\geq 0, j\geq 0$, and $m'\geq 0$

$$\lim_{\gamma\to 0^+} \gamma^{\mu_{m,j}}\left|\int_{-\infty}^{\infty} dv \sum_s (\kappa^{(s)})^{m'} h_{m,j}^{(s)}(v)\right|<\infty, \tag{33}$$

where $\mu_{m,j}=J_{m,j}+1$ with $J_{m,j}\equiv(2m+5j-3)+4\delta_{m,0}+5\delta_{m,1}$ in the generic case defined by $c_1(0)\neq 0$. For the special cases of fixed ions or flat ion distributions, Eq. (33) holds with exponent $\mu_{m,j}=J_{m,j}-j-\delta_{m,1}$. This estimate of the singularity in $h_{m,j}^{(s)}(v)$ is applied in the following section to analyze the resonant particle distributions.

The first bound [Eq. (32)] determines the scaling exponent for A. When Eqs. (18), (21), and (23) are combined we obtain an amplitude equation

$$\dot{A}=\lambda A+\sum_{j=1}^{\infty} p_j|A|^{2j}A, \tag{34}$$

where each nonlinear term has a singular coefficient and the equation is ill-defined as $\gamma\to 0^+$. The cure is to rescale the amplitude

$$A(t)\equiv\gamma^{\beta}r(\gamma t)e^{-i\theta(t)}, \tag{35}$$

with $\beta=\frac{5}{2}$ for the typical case $(c_1(0)\neq 0)$ and in the special cases with $c_1(0)=0$ we require $\beta=2$. Once this is done, the equations for $r(\tau)$ and $\theta(t)$ are nonsingular in the regime of weak growth rates; additional details may be found in Ref. 3.

III. DISTRIBUTION FUNCTIONS AND ELECTRIC FIELD

The scaling [Eq. (35)] of the amplitude has immediate implications for the asymptotic structure of the distributions. From Eq. (17) the Fourier coefficients of $f=F-F_0$ may be written in terms of $r(\tau)$ and $\theta(t)$ in Eq. (35) and h_m

$$\begin{aligned} f_0(v,t)&=\gamma^{2\beta}r(\tau)^2h_0(v,\gamma^{2\beta}r^2),\\ f_k(v,t)&=\gamma^{\beta}r(\tau)e^{-i\theta(t)}[\psi(v)+\gamma^{2\beta}r(\tau)^2h_1(v,\gamma^{2\beta}r^2)],\\ f_{mk}(v,t)&=\gamma^{m\beta}r(\tau)^me^{-im\theta(t)}h_m(v,\gamma^{2\beta}r^2),\quad m\geq 2. \end{aligned} \tag{36}$$

As $\gamma\to 0^+$, $r(\tau)$ is an $\mathcal{O}(1)$ quantity, thus the asymptotic features of each Fourier component are determined by the explicitly shown factors of γ and the asymptotic form of the functions $\psi(v)$ and $h_m(v,\gamma^{2\beta}r^2)$. The dependence on h_m necessitates a separate consideration of the asymptotic behavior for nonresonant and resonant velocities. In the former regime we assume the distance from the linear phase velocity satisfies $v-v_p=\mathcal{O}(1)$, and the resonant regime corresponds to velocities within a neighborhood of v_p that scales with the growth rate, i.e., $v-v_p=\mathcal{O}(\gamma)$. For resonant velocities, the singularities in $\psi(v)$ and $h_m(v,\gamma^{2\beta}r^2)$ come into play and alter the asymptotic features of the distribution function.

A. Nonresonant velocities

For $v-v_p=\mathcal{O}(1)$, the functions $\psi(v)$ and $h_m(v,\sigma)$ are bounded $\mathcal{O}(1)$ quantities (we use $\sigma=\gamma^{2\beta}r^2$ to emphasize this), and the Fourier components (36) combine to yield

$$\begin{aligned} &(F(x,v,t)-F_0(v))/\gamma^{\beta}\\ &\quad=[r(\tau)e^{-i\theta(t)}\Psi(x,v)+cc]+\gamma^{\beta}r(\tau)^2h_0(v,\sigma)\\ &\qquad+\gamma^{2\beta}r(\tau)^3[e^{-i\theta(t)}h_1(v,\sigma)e^{ikx}+cc]\\ &\qquad+\sum_{m=2}^{\infty}[\gamma^{(m-1)\beta}r(\tau)^me^{-im\theta(t)}h_m(v,\sigma)e^{imkx}+cc]\\ &\quad\approx[r(\tau)e^{-i\theta(t)}\Psi(x,v)+cc]+\mathcal{O}(\gamma^{\beta}). \end{aligned} \tag{37}$$

In words, the nonresonant correction to F_0 scales overall as γ^{β}; the leading piece of this correction simply has the form of the linear wave $\Psi(x,v)$ with nonlinear time dependence determined by the mode amplitude $r(\tau)\exp(-i\theta(t))$.

B. Resonant velocities

For $v-v_p=\mathcal{O}(\gamma)$, the functions $\psi(v)$ and $h_m(v,\sigma)$ typically develop singularities as $\gamma\to 0^+$ and these divergences compete with the explicit factors of γ in Eq. (36) to determine the asymptotic form of the distribution. The analysis is simplified by the fact that all relevant singularities are poles of the form described in Eq. (31), and these may be rewritten as a singular factor multiplying a nonsingular function of the rescaled velocity variable $u\equiv(v-v_p)/\gamma$, e.g.

$$\frac{1}{(v-\alpha)^n}=\frac{1}{\gamma^n}\,\frac{1}{(u-i(\zeta+1)/k)^n}. \tag{38}$$

Once this is done, the functions $\psi(v)$ and $h_m(v,\sigma)$, expressed in terms of u, may be substituted into (36); the variable u provides a uniform velocity coordinate for the resonant region.

The puzzle is to deduce the correct overall factor of $1/\gamma^n$ for each function. For $h_m(v,\sigma)$ we have the integral bound (33) on the expansion coefficients which may be rewritten in terms of u

$$\lim_{\gamma\to 0^+}\left|\int_{-\infty}^{\infty} du\sum_s (\kappa^{(s)})^{m'}\gamma^{1+\mu_{m,j}}h_{m,j}(v_p+\gamma u)\right|<\infty. \tag{39}$$

Since all singularities are poles we know the integrand does not have an integrable singularity, hence we conclude that $\gamma^{1+\mu_{m,j}}h_{m,j}(v_p+\gamma u)$ defines a nonsingular function of u

$$h_{m,j}(v_p+\gamma u)\equiv\gamma^{-(1+\mu_{m,j})}\hat{h}_{m,j}(u,\gamma). \tag{40}$$

The nonsingular character of $\hat{h}_{m,j}(u,\gamma)$ can be checked directly for the specific examples in Eqs. (25) and (26), and also verified, in general, from the recursion relations. From the expansion of $h_m(v,\sigma)$, we thus find

$$h_m(v_p+\gamma u,\sigma)=\sum_{j=0}^{\infty}\gamma^{2j\beta-(1+\mu_{m,j})}\hat{h}_{m,j}(u,\gamma)r^{2j}. \tag{41}$$

In the generic case with $\beta=\frac{5}{2}$ and $\mu_{m,j}=J_{m,j}+1$ this gives

$$h_m(v_p+\gamma u,\sigma)=\frac{1}{\gamma^{\delta_m}}\sum_{j=0}^{\infty}\hat{h}_{m,j}(u,\gamma)r^{2j}, \tag{42}$$

where

$$\delta_m=\begin{cases}3 & m=0\\ 6 & m=1 \quad (c_1(0)\neq 0)\\ 2m-1 & m\geqslant 2\end{cases}. \tag{43}$$

In the special cases with fixed ions or flat distributions, then $\beta=2$ and $\mu_{m,j}=J_{m,j}-j-\delta_{m,1}$, and Eq. (42) holds with exponent

$$\delta_m=\begin{cases}2 & m=0\\ 4 & m=1 \qquad (c_1(0)=0).\\ 2m-2 & m\geqslant 2\end{cases} \tag{44}$$

In all cases, we define the nonsingular function $\hat{h}_m(u,r^2,\gamma)\equiv\Sigma_{j=0}^{\infty}\hat{h}_{m,j}(u,\gamma)r^{2j}$ and rewrite Eq. (42)

$$h_m(v_p+\gamma u,\sigma)=\frac{\hat{h}_m(u,r^2,\gamma)}{\gamma^{\delta_m}}. \tag{45}$$

This expression provides the needed information on the singularity of $h(v,\sigma)$ in the resonant region.

It is simpler to obtain the corresponding factorization of the eigenfunction; from the definition (16) we have

$$\psi(v_p+\gamma u)=\frac{1}{\gamma}\left(-\frac{\kappa\cdot\eta_k(v_p+\gamma u)}{u-i/k}\right), \tag{46}$$

and the only subtlety concerns $\eta_k(v_p+\gamma u)$ which is $\mathcal{O}(1)$ in the generic case $(c_1(0)\neq 0)$ and $\mathcal{O}(\gamma)$ in the two special cases with $c_1(0)=0$. Thus we define the nonsingular function $\hat{\psi}(u,\gamma)$ by

$$\psi(v_p+\gamma u)=\frac{\hat{\psi}(u,\gamma)}{\gamma},\quad (c_1(0)\neq 0), \tag{47}$$

in the generic case, but in the special cases the eigenfunction is itself nonsingular and we have

$$\psi(v_p+\gamma u)=\hat{\psi}(u,\gamma),\quad (c_1(0)=0). \tag{48}$$

We are now able to describe the asymptotic structure of the resonant particle distributions.

1. Generic instability: $c_1(0)\neq 0$

For the generic case, inserting Eqs. (45) and (47) into Eq. (36) yields

$$\begin{aligned}&[F(x,v_p+\gamma u,t)-F_0(v_p+\gamma u)]/\gamma^{3/2}\\ &=\{r(\tau)e^{-i\theta(t)}e^{ikx}[\hat{\psi}(u,\gamma)+r(\tau)^2\hat{h}_1(u,r^2,\gamma)]+cc\}\\ &\quad+\sqrt{\gamma}\Bigg\{r(\tau)^2\hat{h}_0(u,r^2,\gamma)+\sum_{m=2}^{\infty}[\gamma^{(m-2)/2}e^{imkx}r(\tau)^m\\ &\quad\times e^{-im\theta(t)}\hat{h}_m(u,r^2,\gamma)+cc]\Bigg\},\end{aligned} \tag{49}$$

neglecting the subdominant terms this gives

$$\begin{aligned}&[F(x,v_p+\gamma u,t)-F_0(v_p+\gamma u)]/\gamma^{3/2}\\ &=\{r(\tau)e^{-i\theta(t)}e^{ikx}[\hat{\psi}(u,\gamma)+r(\tau)^2\hat{h}_1(u,r^2,\gamma)]+cc\}\\ &\quad+\mathcal{O}(\sqrt{\gamma}).\end{aligned} \tag{50}$$

The generic resonant correction to F_0, expressed in the velocity coordinate u, scales overall as $\gamma^{3/2}$; the leading term in this correction has the wavelength of the linear wave but the velocity dependence, $\hat{\psi}(u,\gamma)+r(\tau)^2\hat{h}_1(u,r^2,\gamma)$, is not simply given by the linear eigenfunction. The second term in an $\mathcal{O}(1)$ time-dependent nonlinear contribution. The time dependence is determined by the mode amplitude $r(\tau)\exp(-i\theta(t))$ but the dependence on r is rather complicated.

2. Special cases: $c_1(0)=0$

For the special cases, defined by fixed ions or flat ion distributions, we apply (48), (44), and (45) to Eq. (36) and obtain

$$\begin{aligned}&[F(x,v_p+\gamma u,t)-F_0(v_p+\gamma u)]/\gamma^{2}\\ &=\{r(\tau)e^{-i\theta(t)}e^{ikx}[\hat{\psi}(u,\gamma)+r(\tau)^2\hat{h}_1(u,r^2,\gamma)]+cc\}\\ &\quad+r(\tau)^2\hat{h}_0(u,r^2,\gamma)+\sum_{m=2}^{\infty}[e^{imkx}r(\tau)^m\\ &\quad\times e^{-im\theta(t)}\hat{h}_m(u,r^2,\gamma)+cc].\end{aligned} \tag{51}$$

This is a qualitatively different structure in contrast to Eq. (49); now the resonant correction is $\mathcal{O}(\gamma^2)$ and *all* wavelengths are present at leading order. Thus the spatial dependence is very rich and bears no special relation to the linear instability; a similar observation holds for the dependence on velocity.

C. Electric field

The Fourier components of E are given by Eq. (36) and Poisson's equation

$$\begin{aligned}ikE_k(t)&=\gamma^{\beta}r(\tau)e^{i-i\theta(t)}\Bigg[1+\gamma^{2\beta}r(\tau)^2\\ &\quad\times\int_{-\infty}^{\infty}dv\sum_s h_1^{(s)}(v,\sigma)\Bigg],\\ imkE_{mk}(t)&=\gamma^{m\beta}r(\tau)^m e^{-im\theta(t)}\int_{-\infty}^{\infty}dv\sum_s h_m^{(s)}(v,\sigma),\\ &\qquad m\geqslant 2.\end{aligned} \tag{52}$$

Bounds on the asymptotic form of the integrals can be inferred from the expansion $h_m=\Sigma_j h_{m,j}\sigma^j$ and the bound (33) on the integrals of $h_{m,j}$ (for $m'=0$). The details depend on whether we consider the generic instability or the special cases.

1. Generic instability: $c_1(0)\neq 0$

With Eq. (33) we find

$$\int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v,\sigma)$$

$$=\sum_{j=0}^{\infty} \frac{r^{2j}}{\gamma^{\mu_{m,j}-2\beta j}}\left[\gamma^{\mu_{m,j}}\int_{-\infty}^{\infty} dv \sum_s h_{m,j}^{(s)}(v)\right], \tag{53}$$

where the bracketed integral is an $\mathcal{O}(1)$ quantity from Eq. (33). For the generic instability, $\mu_{m,j}-2\beta j=2m-2+4\delta_{m,0}+5\delta_{m,1}$, i.e., the j dependence cancels, and we obtain an overall scaling for the integral of $h_m^{(s)}(v,\sigma)$

$$\gamma^{\alpha_m}\int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v,\sigma)\sim\mathcal{O}(1), \tag{54}$$

with

$$\alpha_m=\begin{cases} 2 & m=0 \\ 5 & m=1 \\ 2m-2 & m\geqslant 2 \end{cases} \qquad (c_1(0)\neq 0). \tag{55}$$

Hence, with $\beta=\frac{5}{2}$, the generic components are

$$ikE_k(t)=\gamma^{5/2}r(\tau)e^{-i\theta(t)}\left[1+r(\tau)^2\gamma^5\int_{-\infty}^{\infty} dv \sum_s h_1^{(s)}(v,\sigma)\right],$$

$$imkE_{mk}(t)=\gamma^{2+m/2}r(\tau)^m e^{-im\theta(t)}\gamma^{2m-2}\int_{-\infty}^{\infty} dv \times\sum_s h_m^{(s)}(v,\sigma),$$

$$m\geqslant 2. \tag{56}$$

The asymptotic electric field is

$$\frac{E(x,t)}{\gamma^{5/2}}=\frac{1}{k}\left\{-ir(\tau)e^{-i\theta(t)}\left[1+r(\tau)^2\gamma^5 \times\int_{-\infty}^{\infty} dv \sum_s h_1^{(s)}(v,\sigma)\right]e^{ikx}+cc\right\}+\mathcal{O}(\gamma^{1/2}), \tag{57}$$

clearly E is dominated by the wave number of the unstable mode with an overall scaling of $\gamma^{5/2}$. The term $\gamma^5\int dv\,\Sigma_s h_1^{(s)}$ is treated as an $\mathcal{O}(1)$ contribution in light of the estimate (54) above.

2. Special cases: $c_1(0)=0$

For instabilities with fixed ions or flat ion distributions, we have $\beta=2$ and $\mu_{m,j}=J_{m,j}-j-\delta_{m,1}$ in Eq. (33); applying this bound to Eq. (53) yields a new scaling behavior for the integral

$$\gamma^{\alpha_m}\int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v,\sigma)\sim\mathcal{O}(1), \tag{58}$$

with

$$\alpha_m=\begin{cases} 1 & m=0 \\ 3 & m=1 \\ 2m=-3 & m\geqslant 2 \end{cases} \qquad (c_1(0)=0). \tag{59}$$

Now the general expressions for the components reduce to

$$ikE_k(t)=\gamma^2 r(\tau)e^{-i\theta(t)}\left[1+\gamma r(\tau)^2\gamma^3 \times\int_{-\infty}^{\infty} dv \sum_s h_1^{(s)}(v,\sigma)\right],$$

$$imkE_{mk}(t)=\gamma^3 r(\tau)^m e^{-im\theta(t)}\gamma^{2m-3}\int_{-\infty}^{\infty} dv \sum_s h_m^{(s)}(v,\sigma),$$

$$m\geqslant 2, \tag{60}$$

and the asymptotic electric field has the form

$$\frac{E(x,t)}{\gamma^2}=\frac{1}{k}\{-ir(\tau)e^{-i\theta(t)}[1+\mathcal{O}(\gamma)]e^{ikx}+cc\}+\mathcal{O}(\gamma). \tag{61}$$

The overall scaling is now the well known γ^2 or ''trapping scaling'' and the leading term has a much simpler structure. Again we find the wave number k of the linear instability; however, now the time dependence is simply given by the mode amplitude $r(\tau)\exp(-i\theta(t))$.

IV. DISCUSSION

The single wave model, derived originally by O'Neil, Winfrey, and Malmberg, described the interaction of a cold electron beam interacting with a plasma of mobile electrons and fixed ions. In their problem, the infinite extent of the plasma allowed for continuous wave numbers and the dispersion relation for a cold beam was required to select a single wave number corresponding to the maximum growth rate. This wave number characterizes the electric field whose nonlinear time development results from the coupling to resonant particles. The nonresonant plasma simply provides a linear dielectric which supports the wave.

By contrast, we pose a more general problem, allowing for multiple mobile species and not restricting the type of electrostatic instability, but for a finite plasma with periodic boundary conditions. Within this setting, we consider equilibria supporting a single unstable mode and derive the resulting equations for the electric field and distributions in the limit of weak instability. In this asymptotic limit, the physical picture of the original single wave model emerges quite generally. The monochromatic electric field is coupled to the resonant particles and evolves nonlinearly while the nonresonant particles show only a linear response to the electric field.

The amplitude expansions, whose singularity structure form the basis of our analysis, do not provide a practical tool for solving the single wave model. For this purpose, it is more convenient to assume the simplifications of the single wave picture and derive model equations directly from the original Vlasov theory. This development will be presented in a forthcoming paper.

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