

# Amplitude equations for electrostatic waves: Multiple species

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The amplitude equation for an unstable electrostatic wave is analyzed using an expansion in the mode amplitude  $A(t)$ . In the limit of weak instability, i.e.,  $\gamma \rightarrow 0^+$  where  $\gamma$  is the linear growth rate, the nonlinear coefficients are singular and their singularities predict the dependence of  $A(t)$  on  $\gamma$ . Generically the scaling  $|A(t)| = \gamma^{5/2} r(\gamma t)$  as  $\gamma \rightarrow 0^+$  is required to cancel the coefficient singularities to all orders. This result predicts the electric field scaling  $|E_k| \sim \gamma^{5/2}$  will hold universally for these instabilities (including beam-plasma and two-stream configurations) throughout the dynamical evolution and in the time-asymptotic state. In exceptional cases, such as infinitely massive ions, the coefficients are less singular and the more familiar trapping scaling  $|E_k| \sim \gamma^2$  is recovered. © 1998 American Institute of Physics. [S0022-2488(98)03009-6]

## I. INTRODUCTION

### A. Overview and background

We recently presented a detailed analysis of the amplitude equation for an unstable electrostatic mode in an unmagnetized Vlasov plasma [henceforth (I)].<sup>1</sup> In that work, the amplitude equation was studied using an expansion in the wave amplitude, and only the electron motion was considered with the ions treated as a fixed neutralizing background. In this paper we study the effect of including mobile ions and investigate the mode amplitude equation for an electrostatic wave in a multispecies unmagnetized plasma. As in (I), we focus on the regime of weak instability and carefully analyze the structure of the amplitude equation in the limit  $\gamma \rightarrow 0^+$  where  $\gamma$  is the linear growth rate for the unstable wave.

Incorporating the effects of finite ion inertia is a significant step for several reasons. With finite mass, the ion motion occurs on a finite time scale  $T_i < \infty$  and in the limit of very weak growth rates this will be a fast time scale, i.e.,  $T_i \ll \gamma^{-1}$  as  $\gamma \rightarrow 0^+$ . Over the duration of the instability, the ions can readily respond to the electric field of the wave especially resonant ions near the phase velocity. This ion response is always present even for high frequency modes such as plasma waves since at the phase velocity the wave frequency is Doppler-shifted to zero. In addition, there are modes such as ion-acoustic waves that depend on ion density oscillations and are suppressed when the ions are treated as fixed. Allowing for mobile ions is a prerequisite, if we seek the amplitude equation for an unstable ion-acoustic mode.

This paper can be read independently, but the reader is referred to (I) for a discussion of the history of the problem and additional introductory material. An essential difference between our approach and previous work lies in the choice of unperturbed state. Earlier theories assumed an equilibrium with a neutrally stable mode and obtained ill-defined expansion coefficients.<sup>2-6</sup> This can be avoided by taking the weakly unstable equilibrium as the unperturbed state; a choice that naturally leads one to work with the unstable manifold. A brief summary of our results on the multispecies case has appeared elsewhere.<sup>7</sup>

In the limit  $\gamma \rightarrow 0^+$ , the amplitude equation defines a kind of singular perturbation problem whose detailed features reveal asymptotic scaling behavior of the nonlinear wave. This is a key idea behind our approach and it can be formulated more precisely as follows. The mode eigen-

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value  $\lambda = \gamma - i\omega$  can be complex (beam-plasma) or real (two-stream), but in either case the equations for the mode amplitude  $A(t) = \rho(t)e^{-i\theta(t)}$  have the form

$$\dot{\rho} = \gamma\rho + a_1\rho^3 + a_2\rho^5 + \mathcal{O}(\rho^7), \tag{1}$$

$$\dot{\theta} = \omega + a'_1\rho^2 + a'_2\rho^4 + \mathcal{O}(\rho^6). \tag{2}$$

Since the Vlasov equilibrium is assumed to be spatially homogeneous, the evolution of  $\rho(t)$  decouples from the phase  $\theta(t)$ . Suppose for simplicity that the coefficients  $a_j$  are constant (ignoring their dependence on  $\gamma$ ), then we may view (1) as a singular perturbation problem with the linear term  $\gamma\rho$  representing the perturbation. If  $\gamma > 0$ , then there is always a neighborhood of the equilibrium  $\rho = 0$  where the perturbation dominates the unperturbed system and can completely change the dynamics.

As in other such singular problems, a possible strategy is to seek a (singular) change of variables which transforms (1) into a regular perturbation problem. Thus it is natural to define a new amplitude  $r(\tau)$  by

$$\rho(t) = \gamma^\beta r(\gamma^\delta t) \tag{3}$$

and rewrite (1) in terms of  $r(\tau)$

$$\frac{dr}{d\tau} = \gamma^{1-\delta} r(\tau) + \gamma^{2\beta-\delta} a_1(\gamma) r^3 + \gamma^{4\beta-\delta} a_2(\gamma) r^5 + \dots \tag{4}$$

If possible, the choice of  $\beta$  and  $\delta$  should be made so that (i) each term in (4) is well behaved as  $\gamma \rightarrow 0^+$  and (ii) the effect of  $\gamma > 0$  is a regular perturbation of the system at  $\gamma = 0$ . The latter condition requires  $\delta = 1$ ; otherwise the linear term will continue to be a singular perturbation. The exponent  $\beta$  then needs to be chosen (if possible) to achieve a balance between the nonlinear terms and the linear term. In the simplest situation, when the coefficients  $a_1, a_2, \dots$  have well-defined *finite* limits as  $\gamma \rightarrow 0$ , then one has the exponent  $\beta = 1/2$  and this yields

$$\frac{dr}{d\tau} = r(\tau) + a_1(\gamma) r^3 + \gamma a_2(\gamma) r^5 + \dots \tag{5}$$

The unperturbed system  $\dot{r} = r + a_1(0)r^3$  now includes the linear term and near  $r = 0$  the effect of small  $\gamma$  is unimportant. The change of variables (3) is singular at  $\gamma = 0$  as expected, but nevertheless we have determined the overall scaling  $\beta = 1/2$  for nonlinear solutions near the equilibrium in the regime of weak instability. In addition, since terms in (4) at fifth order or higher are at least of order  $\mathcal{O}(\gamma)$ , the unperturbed problem truncates to a simple balance between the linear instability and the dominant nonlinear term.

For an unstable electrostatic wave, the situation is quite different because the nonlinear coefficients  $a_j$  blow up as  $\gamma \rightarrow 0$  and the exponent  $\beta = 1/2$  does not yield a well-behaved equation for  $r(\tau)$ . We attack the problem of finding a satisfactory value for  $\beta$  in two steps. First, we determine what choice for  $\beta$  will control the singularity of the cubic coefficient  $a_1$  in (1), and then we investigate whether this choice will also remove the singularities of the higher order nonlinear terms. Typically, the cubic coefficient diverges asymptotically like  $\gamma^{-4}$  in which case we must choose  $\beta \geq 5/2$  in order to obtain a finite cubic term in the rescaled amplitude equation (4).

Our analysis shows that setting  $\beta = 5/2$  yields a theory that is finite to all orders as  $\gamma \rightarrow 0$  at  $\gamma = 0$  retains (rescaled) nonlinear terms that balance the linear instability. Choosing a larger exponent  $\beta = 5/2 + \epsilon$  would also yield a finite theory to all orders, but now the cubic term is of order  $\mathcal{O}(\gamma^{2\epsilon})$  and all the higher order terms ( $j > 1$ ) are of order  $\mathcal{O}(\gamma^{2\epsilon j})$ ; there is no balance between the linear and nonlinear terms at  $\gamma = 0$ . For  $\epsilon > 0$ , the larger exponent suppresses nonlinear effects too strongly. In this sense, the scaling

$$\rho(t) = \gamma^{5/2} r(\gamma t) \tag{6}$$

is uniquely determined. Of the two steps, the second is by far the most difficult; it is relatively straightforward to compute  $a_1$  and determine its divergence (Sec. III). Verifying that a given choice of  $\beta$  works to all orders requires a detailed study of the recursion relations of the theory and the final theorem is proved by induction (Sec. IV).

In special cases, the cubic coefficient is less singular, diverging like  $\gamma^{-3}$  instead of  $\gamma^{-4}$ . The limit of infinitely massive ions is the best known example of this situation, and in (I) we found that  $\beta=2$  was the scaling needed to make the nonlinear amplitude equation finite to all orders. In this paper, we identify two additional examples where the cubic coefficient has a  $\gamma^{-3}$  singularity: if the effects of the resonant ions are negligible (in a sense we make precise in Sec. III D) or if the electron and ion masses are equal, i.e., for an electron-positron plasma. In the former case we prove that the exponent  $\beta=2$  is correct, but the precise value of  $\beta$  for the electron-positron system is not unambiguously determined due to special complications in the singularity structure of the amplitude expansion.

**B. Notation**

Our model is a one-dimensional, multispecies Vlasov plasma defined by

$$\frac{\partial F^{(s)}}{\partial t} + v \frac{\partial F^{(s)}}{\partial x} + \kappa^{(s)} E \frac{\partial F^{(s)}}{\partial v} = 0, \tag{7}$$

$$\frac{\partial E}{\partial x} = \sum_s \int_{-\infty}^{\infty} dv F^{(s)}(x, v, t). \tag{8}$$

Here  $x$ ,  $t$  and  $v$  are measured in units of  $u/\omega_e$ ,  $\omega_e^{-1}$  and  $u$ , respectively, where  $u$  is a chosen velocity scale and  $\omega_e^2 = 4\pi e^2 n_e / m_e$ . The plasma length is  $L$  with periodic boundary conditions and we adopt the normalization

$$\int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv F^{(s)}(x, v, t) = \left( \frac{z_s n_s}{n_e} \right) L, \tag{9}$$

where  $q_s = ez_s$  is the charge of species  $s$  and  $\kappa^{(s)} \equiv q_s m_e / em_s$ . Note that  $\kappa^{(e)} = -1$  for electrons and that the normalization (9) for negative species makes the distribution function negative.

Let  $F_0(v)$  and  $f(x, v, t)$  denote the multicomponent fields for the equilibrium and perturbation, respectively, and  $\kappa$  the matrix of mass ratios,

$$f \equiv \begin{pmatrix} f^{(s_1)} \\ f^{(s_2)} \\ \vdots \end{pmatrix}, \quad F_0 \equiv \begin{pmatrix} F_0^{(s_1)} \\ F_0^{(s_2)} \\ \vdots \end{pmatrix}, \quad \kappa \equiv \begin{pmatrix} \kappa^{(s_1)} & 0 & 0 & \cdots \\ 0 & \kappa^{(s_2)} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{10}$$

then the system (7) and (8) can be concisely expressed as

$$\frac{\partial f}{\partial t} = \mathcal{L}f + \mathcal{N}(f), \tag{11}$$

where the linear operator is defined by

$$\mathcal{L}f = \sum_{l=-\infty}^{\infty} e^{ilx} (L_l f_l)(v), \tag{12}$$

$$(L_l f_l)(v) = \begin{cases} 0 & l=0 \\ -il[vf_l(v) + \kappa \cdot \eta_l(v) \sum_{s'} \int_{-\infty}^{\infty} dv' f_l^{(s')}(v')] & l \neq 0, \end{cases} \tag{13}$$

with  $\eta_l(v) \equiv -\partial_v F_0 / l^2$ , and the nonlinear operator  $\mathcal{N}$  is

$$\mathcal{N}(f) = \sum_{m=-\infty}^{\infty} e^{imx} \sum_{l=-\infty}^{\infty} \frac{i}{l} \left( \kappa \cdot \frac{\partial f_{m-l}}{\partial v} \right) \sum_{s'} \int_{-\infty}^{\infty} dv' f_l^{(s')}(v'). \tag{14}$$

In the spatial Fourier expansion (12),  $l$  denotes an integer multiple of the basic wave number  $2\pi/L$ , and a primed summation as in (14) omits the  $l=0$  term. The notation  $\kappa \cdot \eta_l(v)$  or  $\kappa \cdot \partial_v f_{m-l}$  denotes matrix multiplication.

An equilibrium  $F_0(v)$  depends on parameters such as the densities or temperatures of the various species, but this dependence will be suppressed unless it is explicitly needed.

Symmetries of the model (11) and the equilibrium  $F_0(v)$  are important qualitative features of the problem, and play a significant role in Sec. II when we formulate the amplitude expansions. Spatial translation,  $\mathcal{T}_a: (x, v) \rightarrow (x + a, v)$ , and reflection,  $\mathcal{R}: (x, v) \rightarrow (-x, -v)$ , act as operators on  $f(x, v, t)$  in the usual way: if  $\alpha$  denotes an arbitrary transformation then  $(\alpha f)(x, v) \equiv f(\alpha^{-1}(x, v))$  where  $(\alpha f)(x, v)$  denotes the transformed distribution function. The operators  $\mathcal{L}$  and  $\mathcal{N}$  commute with  $\mathcal{T}_a$  due to the spatial homogeneity of  $F_0$ , and if  $F_0(v) = F_0(-v)$ , then  $\mathcal{L}$  and  $\mathcal{N}$  also commute with the reflection operator  $\mathcal{R}$ . With periodic boundary conditions,  $x$  is a periodic coordinate so  $\mathcal{T}_a$  and  $\mathcal{R}$  generate  $O(2)$ , the symmetry group of the circle. If only the translation symmetry is present, then the group is  $SO(2)$ .

An inner product is needed in Sec. II to derive the amplitude equation. For two multicomponent fields of  $(x, v)$ , e.g.,  $B = (B^{(s_1)}, B^{(s_2)}, B^{(s_3)}, \dots)$  and  $D = (D^{(s_1)}, D^{(s_2)}, D^{(s_3)}, \dots)$ , we define their inner product by

$$(B, D) \equiv \sum_s \int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dv B^{(s)}(x, v) * D^{(s)}(x, v) = \int_{-L/2}^{L/2} dx \langle B, D \rangle, \tag{15}$$

where

$$\langle B, D \rangle \equiv \sum_s \int_{-\infty}^{\infty} dv B^{(s)}(x, v) * D^{(s)}(x, v). \tag{16}$$

### C. Summary of linear theory

The spectral theory for  $\mathcal{L}$  is well established, and the needed results are simply recalled to establish our notation.<sup>8-10</sup> The eigenvalues  $\lambda = -ilz$  of  $\mathcal{L}$  are determined by the roots  $\Lambda_l(z) = 0$  of the ‘‘spectral function,’’

$$\Lambda_l(z) \equiv 1 + \int_{-\infty}^{\infty} dv \frac{\sum_s \kappa^{(s)} \eta_l^{(s)}(v)}{v - z}. \tag{17}$$

If the contour in (17) is replaced by the Landau contour for  $\text{Im}(z) < 0$  then we have the linear dielectric  $\epsilon_l(z)$ ; for  $\text{Im}(z) > 0$ ,  $\Lambda_l(z)$  and  $\epsilon_l(z)$  are the same function. For  $SO(2)$  symmetry, the eigenvalues are generically complex; for  $O(2)$  symmetry, when the equilibrium is also reflection-symmetric, then real eigenvalues can occur as in a two-stream instability; see, for example, Ref. 11.

Associated with an eigenvalue  $\lambda = -ilz$  is the multicomponent eigenfunction  $\Psi(x, v) = e^{ilx} \psi(v)$  where

$$\psi(v) = - \frac{\kappa \cdot \eta_l}{v - z}. \tag{18}$$

There is also an associated adjoint eigenfunction  $\tilde{\Psi}(x, v) = e^{ilx} \tilde{\psi}(v)/L$  satisfying  $(\tilde{\Psi}, \Psi) = 1$  with

$$\tilde{\psi}(v) = - \frac{1}{\Lambda_l'(z) * (v - z^*)}. \tag{19}$$

Note that all components of  $\tilde{\psi}(v)$  are the same. The normalization in (19) assumes that the root of  $\Lambda_l(z)$  is simple and is chosen so that  $\langle \tilde{\psi}, \psi \rangle = 1$ . The adjoint determines the projection of  $f(x, v, t)$  onto the eigenvector, and this projection defines the time-dependent amplitude of  $\Psi$ , i.e.,  $A(t) \equiv (\tilde{\Psi}, f)$ .

## II. AMPLITUDE EQUATION ON THE UNSTABLE MANIFOLD

### A. Unstable linear modes

The equilibrium  $F_0(v)$  is assumed to support a ‘‘single’’ unstable mode in the following sense. We shall assume that  $E^u$ , the unstable subspace for  $\mathcal{L}$ , is two dimensional. With translation symmetry and periodic boundary conditions, this is the simplest instability problem that can be posed.

Henceforth, let  $k$  denote the wave number of this unstable mode that is associated with the root  $\Lambda_k(z_0) = 0$  which we assume to be simple, i.e.,  $\Lambda'_k(z_0) \neq 0$ . The corresponding eigenvector is

$$\Psi_1(x, v) = e^{ikx} \psi(v) = e^{ikx} \left( -\frac{\kappa \cdot \eta_k}{v - z_0} \right). \tag{20}$$

The root  $z_0 = v_p + i\gamma/k$  determines the phase velocity  $v_p = \omega/k$  and the growth rate  $\gamma$  of the linear mode as the real and imaginary parts of the eigenvalue  $\lambda = -ikz_0 = \gamma - i\omega$ . From (17) and  $\Lambda_k(z_0) = 0$ , we obtain

$$\Lambda_l(z_0) = (l^2 - k^2)/l^2; \tag{21}$$

this identity will be useful shortly.

When  $F_0(v)$  lacks reflection symmetry, then  $\Psi_1$  typically has a nonzero phase velocity and  $\lambda$  is complex. In this case, the identities  $\Lambda_k(z) = \Lambda_{-k}(z)$  and  $\Lambda_k(z)^* = \Lambda_k(z^*)$  imply three additional modes:  $\Psi_1^*$ ,  $\Psi_2$ , and  $\Psi_2^*$  where

$$\Psi_2(x, v) = e^{ikx} \psi(v)^*. \tag{22}$$

These eigenfunctions correspond to eigenvalues  $\lambda^*$ ,  $-\lambda^*$ , and  $-\lambda$ , respectively, and fill out the eigenvalue quartet characteristic of Hamiltonian systems. In the absence of reflection symmetry, the eigenvalues are typically simple and the unstable subspace is two dimensional.

When  $F_0(v)$  is reflection-symmetric, the eigenvalues may be either real or complex. In either case  $\Psi_j$  and  $\mathcal{R} \cdot \Psi_j$  are linearly independent so the eigenvalues have multiplicity two. Now  $E^u$  is two dimensional only in the case of a real eigenvalue since a multiplicity-two complex conjugate pair implies a four-dimensional unstable subspace; this latter possibility is not considered further.

The components of the distribution function along the unstable eigenvectors  $\Psi_1$  and  $\Psi_1^*$  are identified by writing

$$f(x, v, t) = [A(t)\Psi(x, v) + \text{c.c.}] + S(x, v, t), \tag{23}$$

where  $A(t) = (\tilde{\Psi}, f)$  is the mode amplitude for  $\Psi$  and  $(\tilde{\Psi}, S) = 0$ . In (23), the subscript on  $\Psi_1$  has been dropped, and  $\tilde{\Psi} = \exp(ikx)\tilde{\psi}/L$  is the adjoint function for  $z_0$  from (19).

The action of translation  $\mathcal{T}_a$  and reflection  $\mathcal{R}$  on  $f(x, v, t)$  implies an action by these operators on the variables  $(A, S)$ . From (23) we note that  $\mathcal{T}_a f(x, v, t) = f(x - a, v, t)$  is equivalent to

$$\mathcal{T}_a(A, S(x, v)) = (e^{-ika}A, S(x - a, v)), \tag{24}$$

and reflection  $\mathcal{R}f(x, v, t) = f(-x, -v, t)$  yields

$$\mathcal{R}(A, S(x, v)) = (A^*, S(-x, -v)). \tag{25}$$

When these transformations are symmetries, these relations are important for organizing the amplitude expansions introduced in Sec. III.

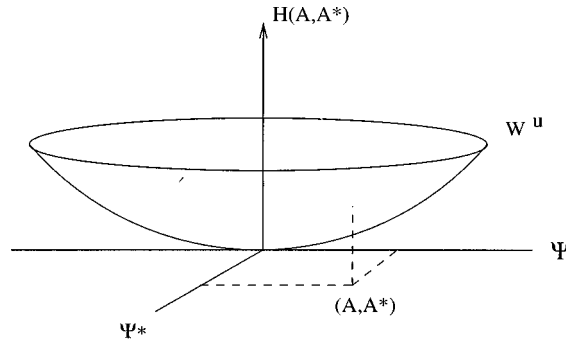


FIG. 1. Schematic representation of the unstable manifold.

**B. Derivation of the amplitude equation**

In the  $(A, S)$  variables, the Vlasov equation (11) becomes:

$$\dot{A} = \lambda A + (\tilde{\Psi}, \mathcal{N}(f)), \tag{26}$$

$$\frac{\partial S}{\partial t} = \mathcal{L}S + \mathcal{N}(f) - [(\tilde{\Psi}, \mathcal{N}(f))\Psi + \text{c.c.}], \tag{27}$$

where

$$(\tilde{\Psi}, \mathcal{N}(f)) = -i \sum_{l=-\infty}^{\infty} \frac{1}{l} \langle \partial_v \tilde{\psi}, \kappa \cdot f_{k-l} \rangle \sum_{s'} \int_{-\infty}^{\infty} dv' f_l^{(s')}(v'). \tag{28}$$

In writing (26) we have used the adjoint relationship  $(\tilde{\Psi}, \mathcal{L}S) = (\mathcal{L}^\dagger \tilde{\Psi}, S) = \lambda(\tilde{\Psi}, S) = 0$  and in (28) an integration by parts shifts the velocity derivative onto  $\tilde{\psi}$ . These coupled equations are equivalent to (11); however, by restricting them to the unstable manifold we obtain an autonomous equation for  $A(t)$ . This procedure is briefly summarized; a more detailed description is provided in (I).

The unstable subspace  $E^u$  is invariant under the linear dynamics  $\partial_t f = \mathcal{L}f$ , but this no longer holds once the nonlinear terms  $\mathcal{N}(f)$  are included. Instead we assume there is a two-dimensional invariant surface, the unstable manifold  $W^u$ , that represents the nonlinear deformation of  $E^u$ . The unstable manifold is tangent to  $E^u$  at the equilibrium, and near  $F_0$  it can be described by a function  $H(x, v, A, A^*)$  which locates the manifold ‘‘above’’ the point  $[A\Psi + A^*\Psi^*]$  in  $E^u$ ; the geometry is illustrated in Fig. 1.

Thus

$$f^u(x, v, t) = [A(t)\Psi(x, v) + \text{c.c.}] + H(x, v, A(t), A^*(t)) \tag{29}$$

represents a distribution function on  $W^u$ . In this expression, the evolution of  $S$  can be determined from  $H$ , i.e.,

$$S(x, v, t) = H(x, v, A(t), A^*(t)) = \begin{pmatrix} H^{(s_1)}(x, v, A(t), A^*(t)) \\ H^{(s_2)}(x, v, A(t), A^*(t)) \\ \vdots \end{pmatrix}. \tag{30}$$

When this representation is substituted into (26) and (27) we obtain

$$\dot{A} = \lambda A + (\tilde{\Psi}, \mathcal{N}(f^u)), \tag{31}$$

$$\frac{\partial S}{\partial t} \Big|_{f^u} = \mathcal{L}H + \mathcal{N}(f^u) - [(\tilde{\Psi}, \mathcal{N}(f^u))\Psi + \text{c.c.}]. \tag{32}$$

Note that (31) defines an *autonomous* two-dimensional flow describing the self-consistent nonlinear evolution of the unstable mode; this is the amplitude equation we shall study.

Certain general features of the amplitude equation follow from the transformations (24) and (25) when these are symmetries of the Vlasov equation (11). For our problem, the amplitude equation always has translation symmetry (24) and we can apply standard results on the form of such symmetric equations. In particular, a two-dimensional vector field  $\dot{A} = V(A, A^*)$  that is symmetric with respect to  $A \rightarrow Ae^{-ika}$  can be written as  $V(A, A^*) = Ap(\sigma)$  where  $\sigma = |A|^2$  and  $p(\sigma)$  is a smooth function determined from  $V$ .<sup>12</sup> Hence we know the right hand side of (31) takes the general form,

$$Ap(\sigma) = \lambda A + (\tilde{\Psi}, \mathcal{N}(f^u)), \tag{33}$$

where the function  $p(\sigma)$  must be determined from the Vlasov equation. Typically  $p(\sigma)$  is complex-valued, however, when  $F_0$  is reflection-symmetric then  $p(\sigma)$  is forced to be real.

**C. Analysis of  $H(x, v, A, A^*)$**

An equation for  $H$  follows by requiring consistency between (30) and (32). Equating the time derivative of (30) with the right hand side of (32) gives

$$\frac{\partial H}{\partial A} \dot{A} + \frac{\partial H}{\partial A^*} \dot{A}^* = \mathcal{L}H + \mathcal{N}(f^u) - [(\tilde{\Psi}, \mathcal{N}(f^u))\Psi + \text{c.c.}], \tag{34}$$

which is to be solved for  $H$  subject to (35). Since the manifold is tangent to  $E^u$  at the equilibrium,  $H$  must satisfy

$$0 = H(x, v, 0, 0) = \frac{\partial H}{\partial A}(x, v, 0, 0) = \frac{\partial H}{\partial A^*}(x, v, 0, 0). \tag{35}$$

The symmetries of the problem impose general restrictions on  $H$  because the unstable manifold is mapped onto itself by a symmetry transformation. For a translation this invariance means that  $f^u(x - a, v, t)$  corresponds to a point on the manifold and thus can be written as,

$$(\mathcal{T}_a f^u)(x, v) = [A\Psi(x - a, v) + \text{c.c.}] + H(x - a, v, A, A^*) \tag{36}$$

$$= [A'\Psi(x, v) + \text{c.c.}] + H(x, v, A', A'^*). \tag{37}$$

Consistency between these two expressions requires that the transformed amplitude  $A'$  is related to the initial amplitude  $A$  by  $A' = e^{-ika}A$  [cf. (24)], and this implies that  $H$  satisfies

$$H(x - a, v, A, A^*) = H(x, v, e^{ika}A, e^{-ika}A^*). \tag{38}$$

If, in addition, the system (26) and (27) has reflection symmetry, then similar reasoning yields

$$H(-x, -v, A, A^*) = H(x, v, A^*, A). \tag{39}$$

These general symmetry properties of  $H$  usefully constrain the form of the Fourier expansion

$$H(x, v, A, A^*) = \sum_{l=-\infty}^{\infty} e^{ilx} H_l(v, A, A^*) = \sum_{l=-\infty}^{\infty} e^{ilx} \begin{pmatrix} H_l^{(s_1)}(v, A, A^*) \\ H_l^{(s_2)}(v, A, A^*) \\ \vdots \end{pmatrix}. \tag{40}$$

Applying (38) shows that the components  $H_l$  must vanish unless  $l$  is an integer multiple of  $k$  and that the nonzero components have the general form

$$H_0(v, A, A^*) = \sigma h_0(v, \sigma), \quad H_k(v, A, A^*) = A\sigma h_1(v, \sigma), \\ H_{mk}(v, A, A^*) = A^m h_m(v, \sigma) \quad \text{for } m \geq 2, \tag{41}$$

where  $H_{-l} = H_l^*$ . The functions  $h_m$  are not determined by symmetry; however, if reflection symmetry also holds, then

$$h_m(-v, \sigma) = h_m(v, \sigma)^* \tag{42}$$

The reasoning leading to these symmetry results can be briefly outlined. The translation symmetry (38) requires

$$e^{-ila} H_l(v, A, A^*) = H_l(v, A e^{-ika}, A^* e^{ika}). \tag{43}$$

By choosing  $a = 2\pi/k$ , this implies we find  $H_l = 0$  unless  $\exp(2\pi i(l/k)) = 1$ , hence the nonzero components satisfy  $l = mk$  for integer  $m$ . Now recall that a function  $G(v, A, A^*)$  which is invariant under  $A \rightarrow A e^{-ika}$  has the form  $G(v, A, A^*) = g(v, \sigma)$  where  $\sigma = |A|^2$  and  $g(v, \sigma)$  is a smooth function determined by  $G$ . The relation in (43) implies that  $G(v, A, A^*) = (A^*)^m H_{mk}(v, A, A^*)$  is such an invariant function. The factor  $(A^*)^m$  in  $G$  requires that the corresponding  $g(v, \sigma)$  contain an overall factor of  $\sigma^m$ ; hence we set  $g(v, \sigma) = \sigma^m h(v, \sigma)$ . The resulting expression,  $(A^*)^m H_{mk} = \sigma^m h(v, \sigma)$ , then implies  $H_{mk} = A^m h(v, \sigma)$ . This general form, together with the requirement (35), leads to general expressions for the Fourier components of  $H$  in (41).

**D. Equations for  $p(\sigma)$  and  $h_m(v, \sigma)$**

The calculation of the amplitude equation (31) reduces to a determination of the function  $p(\sigma)$  in (33); this requires the evaluation of  $(\tilde{\Psi}, \mathcal{N}(f^u))$  from (28) and involves the Fourier components of  $f^u$

$$f_l^u = [A \psi(v) \delta_{l,k} + A^* \psi(v)^* \delta_{l,-k}] + H_l(v, A, A^*). \tag{44}$$

Combining (28) with (41) and (44) yields

$$\begin{aligned} (\tilde{\Psi}, \mathcal{N}(f^u)) = & \frac{-iA\sigma}{k} \left\{ \langle \partial_v \tilde{\psi}, \kappa \cdot (h_0 - h_2) \rangle + \frac{\Gamma_2}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle \right. \\ & + \sigma \left[ \Gamma_1 \langle \partial_v \tilde{\psi}, \kappa \cdot h_0 \rangle - \Gamma_1^* \langle \partial_v \tilde{\psi}, \kappa \cdot h_2 \rangle + \frac{\Gamma_2}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_1^* \rangle - \frac{\Gamma_2^*}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_3 \rangle \right] \\ & \left. + \sum_{l=3}^{\infty} \frac{\sigma^{l-2}}{l} [\Gamma_l \langle \partial_v \tilde{\psi}, \kappa \cdot h_{l-1}^* \rangle - \sigma \Gamma_l^* \langle \partial_v \tilde{\psi}, \kappa \cdot h_{l+1} \rangle] \right\}. \tag{45} \end{aligned}$$

In this expression, sum over the velocity integrals of  $h_m^{(s)}$  is denoted by

$$\Gamma_m(\sigma) \equiv \sum_s \int_{-\infty}^{\infty} dv h_m^{(s)}(v, \sigma). \tag{46}$$

Now comparing (33) and (45) provides the desired expression for  $p$

$$\begin{aligned} p(\sigma) = & \lambda - \frac{i\sigma}{k} \left\{ \langle \partial_v \tilde{\psi}, \kappa \cdot (h_0 - h_2) \rangle + \frac{\Gamma_2}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle + \sigma \left[ \Gamma_1 \langle \partial_v \tilde{\psi}, \kappa \cdot h_0 \rangle - \Gamma_1^* \langle \partial_v \tilde{\psi}, \kappa \cdot h_2 \rangle \right. \right. \\ & + \frac{\Gamma_2}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_1^* \rangle - \frac{\Gamma_2^*}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_3 \rangle \left. \right] + \sum_{l=3}^{\infty} \frac{\sigma^{l-2}}{l} [\Gamma_l \langle \partial_v \tilde{\psi}, \kappa \cdot h_{l-1}^* \rangle \\ & \left. - \sigma \Gamma_l^* \langle \partial_v \tilde{\psi}, \kappa \cdot h_{l+1} \rangle] \right\}. \tag{47} \end{aligned}$$

This expression for  $p$  involves the functions  $h_m(v, \sigma)$  which define  $H$ . The equations that determine  $h_m(v, \sigma)$  follow from the general equation for  $H$  in (34). In presenting these relations we shall use  $\mathcal{P}_\perp$ , the orthogonal projection defined by the unstable mode. Let  $\mathcal{P}$  denote the projection operator onto  $\psi(v)$  of a multicomponent function  $g(v)$ ,

$$(\mathcal{P}g)(v) \equiv \langle \tilde{\psi}, g \rangle \psi(v), \tag{48}$$

with the orthogonal projection  $\mathcal{P}_\perp \equiv I - \mathcal{P}$  given by

$$(\mathcal{P}_\perp g)(v) = g(v) - \psi(v) \langle \tilde{\psi}, g \rangle. \tag{49}$$

With the previous expression for  $(\tilde{\Psi}, \mathcal{N}(f^u))$  in (45) and the notation in (41) for the Fourier components of  $H$ , the components of (34) take the form:

$$(p + p^*) \left[ h_0 + \sigma \frac{\partial h_0}{\partial \sigma} \right] = \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \left\{ \left[ \psi^* + \sigma(h_1^* - \psi \Gamma_1^*) + \sigma^2 h_1^* \Gamma_1 + \sum_{l=2}^{\infty} \frac{\sigma^{l-1}}{l} h_l^* \Gamma_l \right] - \text{c.c.} \right\}, \tag{50}$$

$$\begin{aligned} & \left[ (2p + p^*) h_1 - (L_k h_1) + (p + p^*) \sigma \frac{\partial h_1}{\partial \sigma} \right] \\ &= \frac{i}{k} \mathcal{P}_\perp \frac{\partial}{\partial v} \kappa \cdot \left\{ h_0 - h_2 + \frac{1}{2} \psi^* \Gamma_2 + \sigma \left[ h_0 \Gamma_1 - h_2 \Gamma_1^* f + \frac{1}{2} h_1^* \Gamma_2 - \frac{1}{2} h_3 \Gamma_2^* \right] \right. \\ & \quad \left. + \sum_{l=3}^{\infty} \frac{\sigma^{l-2}}{l} [h_{l-1}^* \Gamma_l - \sigma h_{l+1} \Gamma_l^*] \right\}, \end{aligned} \tag{51}$$

$$\begin{aligned} & \left[ 2p h_2 - L_{2k} h_2 + (p + p^*) \sigma \frac{\partial h_2}{\partial \sigma} \right] \\ &= \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \left\{ \psi + \sigma \left[ h_1 + \psi \Gamma_1 - h_3 + \frac{1}{2} h_0 \Gamma_2 + \frac{1}{3} \psi^* \Gamma_3 \right] \right. \\ & \quad \left. \times \sigma^2 \left[ h_1 \Gamma_1 - h_3 \Gamma_1^* - \frac{1}{2} h_4 \Gamma_2^* + \frac{1}{3} h_1^* \Gamma_3 \right] - \frac{\sigma^3}{3} h_5 \Gamma_3^* + \sum_{l=4}^{\infty} \frac{\sigma^{l-2}}{l} [h_{l-2}^* \Gamma_l - \sigma^2 h_{l+2} \Gamma_l^*] \right\}, \end{aligned} \tag{52}$$

and

$$\begin{aligned} & \left[ m p h_m - L_{mk} h_m + (p + p^*) \sigma \frac{\partial h_m}{\partial \sigma} \right] = \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \left\{ h_{m-1} + \frac{\psi}{m-1} \Gamma_{m-1} + \sum_{l=2}^{m-2} \frac{h_{m-l}}{l} \Gamma_l + \sigma \left[ h_{m-1} \Gamma_1 \right. \right. \\ & \quad \left. \left. - h_{m+1} + \frac{h_1}{m-1} \Gamma_{m-1} + \frac{h_0}{m} a m a_m + \frac{\psi^*}{m+1} \Gamma_{m+1} \right] + \sigma^2 \left[ -h_{m+1} \Gamma_1^* + \frac{h_1^*}{m+1} \Gamma_{m+1} \right] \right. \\ & \quad \left. + \sum_{l=m+2}^{\infty} \frac{\sigma^{l-m}}{l} h_{l-m}^* \Gamma_l - \sum_{l=2}^{\infty} \frac{\sigma^l}{l} h_{m+l} \Gamma_l^* \right\} \end{aligned} \tag{53}$$

for  $m=0,1,2$ , and  $m>2$ , respectively.

Together with (47) these component equations determine the functions  $p(\sigma)$  and  $\{h_m(\sigma)\}_{m=0}^\infty$ ; however, they cannot be solved except using the expansions introduced in the next section. From a practical standpoint, we have achieved a reduction of the problem to the analysis of functions of a *single* real variable, i.e.,  $\sigma$ . In the study of the amplitude equation (31) this reduction represents a very useful simplification.

### III. EXPANSIONS, RECURSION RELATIONS, AND PINCHING SINGULARITIES

The amplitude equation on the unstable manifold,

$$\dot{A} = Ap(\sigma), \tag{54}$$

is analyzed by expressing  $p(\sigma)$ ,  $h_m(v, \sigma)$ , and  $\Gamma_m(\sigma)$  as power series in  $\sigma$ :

$$p(\sigma) = \sum_{j=0}^\infty p_j \sigma^j, \quad h_m(v, \sigma) = \sum_{j=0}^\infty h_{m,j}(v) \sigma^j, \quad \Gamma_m(\sigma) = \sum_{j=0}^\infty \Gamma_{m,j} \sigma^j, \tag{55}$$

where  $\Gamma_{m,j} = \int dv \Sigma_s h_{m,j}^{(s)}(v)$ .

The coefficients  $p_j$  and  $h_{m,j}(v)$  are calculated by recursively solving (47) and (50)–(53). A recursion relation for  $p_j$  follows from (47) by inserting the expansions above and solving at each order in  $\sigma$ ; this calculation yields

$$p_0 = \lambda, \tag{56}$$

$$p_j = \frac{i}{k} [\mathcal{A}_j + \mathcal{B}_j] \quad \text{for } j \geq 1, \tag{57}$$

where

$$\begin{aligned} \mathcal{A}_j = & -\langle \partial_v \tilde{\psi}, \kappa \cdot (h_{0,j-1} - h_{2,j-1}) \rangle - \frac{1}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle \Gamma_{2,j-1} - \sum_{l=0}^{j-2} [\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,j-l-2} \rangle \Gamma_{1,l} \\ & - \langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,j-l-2} \rangle \Gamma_{1,l}^*], \end{aligned} \tag{58}$$

$$\begin{aligned} \mathcal{B}_j = & - \sum_{l=0}^{j-2} \left\{ \frac{1}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,j-l-2}^* \rangle \Gamma_{2,l} + \sum_{m=0}^l \left[ \frac{\Gamma_{j-l+1,m}}{j-l+1} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{j-l-1,m}^* \rangle \right. \right. \\ & \left. \left. - \frac{\Gamma_{j-l,m}^*}{j-l} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{j-l+1,l-m} \rangle \right] \right\}. \end{aligned} \tag{59}$$

Here and below, a summation is understood to be omitted if the lower limit exceeds the upper limit. This organization of the terms in (58) and (59) will turn out to distinguish to different singular behaviors:  $\mathcal{A}_j \sim \gamma^{-(5j-1)}$  and  $\mathcal{B}_j \sim \gamma^{-(5j-2)}$ .

The corresponding relations for  $h_{m,j}(v)$  are determined similarly by applying the expansions (55) to the general equations (50)–(53); the resulting equations are solved for  $h_{m,j}(v)$ . Our results are conveniently stated in terms of the resolvent operator  $R_l(w) \equiv (w - L_l)^{-1}$  and certain auxiliary functions  $I_{m,j}(v)$  whose detailed expressions are provided below. For  $m=0$ ,  $h_{0,j}$  is simply

$$h_{0,j}(v) = \frac{I_{0,j}(v)}{(1+j)(\lambda + \lambda^*)}, \tag{60}$$

and the coefficients  $h_{m,j}$  for  $m > 0$  require the resolvent

$$h_{m,j}(v) = R_{mk}(w_{m,j}) I_{m,j} \tag{61}$$

with the complex numbers  $w_{m,j}$  defined by

$$w_{m,j} \equiv (j + \delta_{m,1})(\lambda + \lambda^*) + m\lambda = 2(j + \delta_{m,1})\gamma + m\lambda. \tag{62}$$

A general expression for  $R_l(w)$  follows from (13) by solving  $(w - L_l)f = g$  for  $f$ .<sup>10,13</sup> Here both  $g(v) = (g^{(s_1)}(v), g^{(s_2)}(v), \dots)$  and  $f$  are multicomponent fields, and  $R_l(w)$  acts by

$$R_l(w)g = \begin{pmatrix} (R_l(w)g)^{(s_1)}(v) \\ (R_l(w)g)^{(s_2)}(v) \\ \vdots \end{pmatrix}, \tag{63}$$

where

$$(R_l(w)g)^{(s)}(v) = \frac{1}{il(v - iw/l)} \left[ g^{(s)}(v) - \frac{\kappa^{(s)} \eta_l^{(s)}}{\Lambda_l(iw/l)} \sum_{s'} \int_{-\infty}^{\infty} dv' \frac{g^{(s')}(v')}{v' - iw/l} \right]. \tag{64}$$

Note the following significant feature: the arguments  $w_{m,j}$  of the resolvent in (61) determine poles of  $h_{m,j}(v)$  located at  $v = z_{m,j}$  where  $z_{m,j} \equiv iw_{m,j}/mk$ . For  $m \geq 1$ , these poles always fall in the upper half-plane above the phase velocity  $v_p$ :

$$z_{m,j} = z_0 + \frac{i\gamma d_{m,j}}{k} = v_p + \frac{i\gamma(1 + d_{m,j})}{k}, \tag{65}$$

where  $d_{m,j} \equiv 2(j + \delta_{m,1})/m$ .

**A. Recursion relations for  $I_{m,j}$**

The auxiliary functions  $I_{0,j}(v)$  are defined by

$$I_{0,0}(v) = \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot (\psi^* - \psi) \tag{66}$$

for  $j=0$ , and for  $j \geq 1$

$$I_{0,j}(v) = - \sum_{n=0}^{j-1} (1+n)(p_{j-n} + p_{j-n}^*) h_{0,n}(v) + \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \left\{ \left[ h_{1,j-1}^* - \psi \Gamma_{1,j-1}^* + \sum_{n=0}^{j-2} h_{1,n}^* \Gamma_{1,j-n-2} \right. \right. \\ \left. \left. + \sum_{n=0}^{j-1} \sum_{l=0}^n \frac{h_{j-n+1,l}^*}{j-n+1} \Gamma_{j-n+1,n-l} \right] - \text{c.c.} \right\}. \tag{67}$$

For  $m=1$ , the functions  $I_{1,j}(v)$  are given by

$$I_{1,j}(v) = - \sum_{n=0}^{j-1} [(2+n)p_{j-n} + (1+n)p_{j-n}^*] h_{1,n} + \frac{i}{k} \mathcal{P}_\perp \frac{\partial}{\partial v} \kappa \cdot \left\{ h_{0,j} - h_{2,j} + \frac{1}{2} \psi^* \Gamma_{2,j} \right. \\ \left. + \sum_{n=0}^{j-1} \left[ h_{0,n} \Gamma_{1,j-n-1} - h_{2,n} \Gamma_{1,j-n-1}^* + \frac{1}{2} h_{1,n}^* \Gamma_{2,j-n-1} - \frac{1}{2} h_{3,n} \Gamma_{2,j-n-1}^* \right. \right. \\ \left. \left. + \sum_{m=0}^n \left( \frac{h_{j-n+1,m}^*}{j-n+2} \Gamma_{j-n+2,n-m} \right) \right] - \sum_{n=0}^{j-2} \sum_{m=0}^n \left[ \frac{h_{j-n+2,m}}{j-n+1} \Gamma_{j-n+1,n-m}^* \right] \right\}. \tag{68}$$

The notation  $\mathcal{P}_\perp g(v)$  was defined in (49):  $\mathcal{P}_\perp g(v) = g(v) - \psi(v) \langle \tilde{\psi}, g \rangle$ .

For  $m=2$  the functions  $I_{2,j}(v)$  are given by

$$I_{2,0}(v) = \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \psi \tag{69}$$

for  $j=0$ , and for  $j \geq 1$

$$\begin{aligned}
 I_{2,j}(v) = & - \sum_{n=0}^{j-1} [(2+n)p_{j-n} + np_{j-n}^*]h_{2,n} + \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \left\{ h_{1,j-1} + \psi \Gamma_{1,j-1} - h_{3,j-1} + \frac{1}{3} \psi^* \Gamma_{3,j-1} \right. \\
 & + \frac{1}{2} \sum_{n=0}^{j-1} h_{0,n} \Gamma_{2,j-n-1} + \sum_{n=0}^{j-2} \left[ h_{1,n} \Gamma_{1,j-n-2} - h_{3,n} \Gamma_{1,j-n-2}^* - \frac{1}{2} h_{4,n} \Gamma_{2,j-n-2}^* \right. \\
 & + \left. \frac{1}{3} h_{1,n}^* \Gamma_{3,j-n-2} + \sum_{m=0}^n \frac{h_{j-n,m}^*}{j-n+2} \Gamma_{j-n+2,n-m} \right] - \sum_{n=0}^{j-3} \frac{h_{5,n}}{3} \Gamma_{3,j-n-3}^* \\
 & \left. - \sum_{n=0}^{j-4} \sum_{m=0}^n \frac{h_{j-n+2,m}}{j-n} \Gamma_{j-n,n-m}^* \right\}. \tag{70}
 \end{aligned}$$

For  $m > 2$ , the functions  $I_{m,j}(v)$  are given by

$$\begin{aligned}
 I_{m,j}(v) = & - \sum_{n=0}^{j-1} [(m+n)p_{j-n} + np_{j-n}^*]h_{m,n}(v) + \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \left\{ h_{m-1,j} + \frac{\psi}{m-1} \Gamma_{m-1,j} \right. \\
 & + \sum_{l=2}^{m-2} \sum_{n=0}^j \frac{h_{m-l,n}}{l} \Gamma_{l,j-n} - h_{m+1,j-1} + \frac{\psi^*}{m+1} \Gamma_{m+1,j-1} + \sum_{n=0}^{j-1} \left[ h_{m-1,n} \Gamma_{1,j-n-1} \right. \\
 & + \left. \frac{h_{1,n}}{m-1} \Gamma_{m-1,j-n-1} + \frac{h_{0,n}}{m} \Gamma_{m,j-n-1} \right] + \sum_{n=0}^{j-2} \left[ -h_{m+1,n} \Gamma_{1,j-n-2}^* + \frac{h_{1,n}^*}{m+1} \Gamma_{m+1,j-n-2} \right. \\
 & \left. \left. + \sum_{l=0}^n \left( \frac{h_{j-n,l}^*}{j+m-n} \Gamma_{j+m-n,n-l} - \frac{h_{m+j-n,l}}{j-n} \Gamma_{j-n,n-l}^* \right) \right] \right\}; \tag{71}
 \end{aligned}$$

in this last expression, if a subscript is negative the term is understood to be omitted, e.g., for  $j=0$ ,  $h_{m+1,j-1}$  is omitted.

**B. Useful identities**

The relations (57), (60) and (61), and (66)–(71) can be applied systematically to calculate  $p_j$  and  $h_{m,j}$  to any order. The leading coefficient  $p_0$  is determined by linear theory (56) and from the linear eigenfunction  $\psi$  one can also calculate  $h_{0,0}$  and  $h_{2,0}$ , c.f. (66) and (69), respectively. These two coefficients then suffice to calculate  $p_1$  from (57). From  $\{p_1, h_{0,0}, h_{2,0}\}$ , the coefficients  $h_{1,0}$  and  $h_{3,0}$  can be determined and then  $h_{0,1}$  and  $h_{2,1}$ . This provides the input to calculate  $p_2$ , and from this point on the structure of the calculation to all orders falls into the simple pattern summarized in Table I.

For the remainder of the paper, our analysis of the expansion (55) is facilitated by several identities which we summarize here. These relations allow  $\Gamma_{m,j}, h_{m,j}$ , and certain important integrals to be obtained very simply from the auxiliary functions  $I_{m,j}$ .

If  $m=0$  we note that (60) and (66) and (67) imply  $\Gamma_{0,j}=0$ . For  $m > 0$ , by substituting (61) into  $\Gamma_{m,j} = \int dv \sum_s h_{m,j}^{(s)}$  and rearranging, we obtain

$$\Gamma_{m,j} = \frac{-i/mk}{\Lambda_{mk}(z_{m,j})} \sum_s \int_{-\infty}^{\infty} dv \frac{I_{m,j}^{(s)}(v)}{v - z_{m,j}} \quad (m > 0). \tag{72}$$

With (72) for  $\Gamma_{m,j}$ , the general form for  $h_{m,j}$  in (61) can be reexpressed as

$$h_{m,j}(v) = \left( \frac{-i}{mk} \right) \frac{I_{m,j}(v)}{v - z_{m,j}} - \left( \frac{\kappa \cdot \eta_{mk}}{v - z_{m,j}} \right) \Gamma_{m,j} \quad (m > 0). \tag{73}$$

In this way  $\Gamma_{m,j}$  and  $h_{m,j}$  are written explicitly in terms of  $I_{m,j}$ .

The coefficients  $p_j$  depend on integrals of the form  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{m,j} \rangle$ , and it is helpful to express these integrals directly in terms of  $I_{m,j}$  also. From (60) we obtain the identity

TABLE I. Order of calculation of  $h_{m,j}(v)$  and  $p_j$  from  $\psi(v)$ . The flow of calculation of the  $h_{m,j}(v)$  is indicated by moving downward. From  $\psi(v)$ ,  $h_{0,0}$  and  $h_{2,0}$  can be calculated and then  $p_1$  determined;  $h_{1,0}$  and  $h_{3,0}$  are calculated next from  $\{p_1, h_{0,0}, h_{2,0}\}$  and then  $h_{0,1}$  and  $h_{2,1}$  can be evaluated. This then determines  $p_2$ , and so forth. For  $j_1 \geq 2$ ,  $p_{j_1}$  requires prior calculation of  $h_{m,j}$  for  $0 \leq m \leq j_1 + 1$  and  $0 \leq j \leq j_1 - m + 1 - 2(\delta_{m,0} + \delta_{m,1})$ .

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	...
$p_0$		$\psi(v)$						
$p_1$	$h_{0,0}$	–	$h_{2,0}$					
$p_2$	$h_{0,1}$	$h_{1,0}$	$h_{2,1}$	$h_{3,0}$				
$p_3$	$h_{0,2}$	$h_{1,1}$	$h_{2,2}$	$h_{3,1}$	$h_{4,0}$			
$p_4$	$h_{0,3}$	$h_{1,2}$	$h_{2,3}$	$h_{3,2}$	$h_{4,1}$	$h_{5,0}$		
$p_5$	$h_{0,4}$	$h_{1,3}$	$h_{2,4}$	$h_{3,3}$	$h_{4,2}$	$h_{5,1}$	$h_{6,0}$	
$\vdots$								

$$\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,j} \rangle = \frac{1}{2\gamma(1+j)\Lambda'_k(z_0)} \int_{-\infty}^{\infty} dv \frac{\sum_s \kappa^{(s)} I_{0,j}^{(s)}}{(v-z_0)^2}; \tag{74}$$

similarly, by integrating (73) we find

$$\langle \partial_v \tilde{\psi}, \kappa \cdot h_{m,j} \rangle = \left( \frac{-i}{mk\Lambda'_k(z_0)} \right) \int_{-\infty}^{\infty} dv \frac{\sum_s \kappa^{(s)} I_{m,j}^{(s)}}{(v-z_0)^2(v-z_{m,j})} - \left( \frac{\Gamma_{m,j}}{\Lambda'_k(z_0)} \right) \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^2 \eta_{mk}^{(s)}}{(v-z_0)^2(v-z_{m,j})}. \tag{75}$$

**C. Analysis of the cubic coefficient**

It is instructive at this point to evaluate and examine the cubic coefficient  $p_1$  in detail. This coefficient illustrates the occurrence of pinching singularities and the procedures needed to evaluate such singular integrals.

From (57) we have

$$p_1 = -\frac{i}{k} \left[ \langle \partial_v \tilde{\psi}, \kappa \cdot (h_{0,0} - h_{2,0}) \rangle + \frac{\Gamma_{2,0}}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle \right] \tag{76}$$

so  $h_{0,0}(v)$ ,  $h_{2,0}(v)$  and  $\Gamma_{2,0}$  are needed to calculate  $p_1$ . From (60) and (66) we have

$$h_{0,0}(v) = -\frac{1}{k^2} \frac{\partial}{\partial v} \left[ \frac{\kappa^2 \cdot \eta_k}{(v-z_0)(v-z_0^*)} \right], \tag{77}$$

where  $\kappa^2$  denotes the square of the mass matrix. From (21), (69) and (72) we obtain

$$\Gamma_{2,0} = -\frac{2}{3k^2} \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^2 \cdot \eta_k)^{(s)}}{(v-z_0)^3}, \tag{78}$$

and inserting (69) into (73) yields

$$h_{2,0}(v) = \frac{1}{2k^2} \left( \frac{\kappa \cdot \partial_v \psi}{v-z_0} \right) - \frac{\Gamma_{2,0}}{4} \left( \frac{\kappa \cdot \eta_k}{v-z_0} \right). \tag{79}$$

Here we have made the substitution  $\eta_{2k} = \eta_k/4$ .

We are concerned with the form of the various integrals in  $p_1$  in the asymptotic regime of small growth rate  $\gamma \rightarrow 0^+$ . The behavior of these integrals is determined by any complex singularities of the integrand that approach the real velocity axis as  $\gamma \rightarrow 0^+$ . We shall always assume that whatever complex analytic structure characterizes  $F_0$  there are never any singularities that move to the real axis in this limit and therefore the singularities in  $\eta_k$  can be regarded as irrelevant in analyzing a given integral.

The singularities of interest are associated with the explicit poles in the denominator of a given integrand. For example,  $\Gamma_{2,0}$  has a third-order pole at  $v = z_0$  which approaches the real axis from above as  $\gamma \rightarrow 0^+$ . In this case, there are no corresponding poles in the lower half-plane, and consequently no pinching singularity develops; therefore  $\Gamma_{2,0}$  has a finite limit as the growth rate goes to zero. Similarly, we see by inspection that  $h_{2,0}$  has a third-order pole at  $v = z_0$  and consequently the integrand of  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0} \rangle$  has a fifth-order pole at  $v = z_0$ . However, again there is no pinching singularity and this integral also has a finite limit as  $\gamma \rightarrow 0^+$ .

The remaining integrals,  $\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle$  and  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle$ , are more interesting. The simplest example is  $\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle$  whose integrand has a double pole at  $v = z_0$  and a simple pole at  $v = z_0^*$ :

$$\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle = -\frac{1}{\Lambda_k'(z_0)} \sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^2 \cdot \eta_k)^{(s)}}{(v - z_0)^2 (v - z_0^*)}. \tag{80}$$

As  $\gamma \rightarrow 0^+$ , these poles approach the real axis from above and below; this creates a pinching singularity and the integral has a singular limit. We analyze this integral in two steps that can be applied to all such singular integrals: a partial fraction expansion extracts the singularity and the dispersion relation is applied to eliminate integrals over the electron distribution function. The motivation for the second step is to incorporate the one relation between the different distribution functions and to expose the effect of treating the ions as mobile.

The partial fraction expansion of the integrand yields

$$\frac{1}{(v - z_0)^2 (v - z_0^*)} = \left(\frac{ik}{2\gamma}\right)^2 \left[ \frac{1}{v - z_0^*} - \frac{1}{v - z_0} \right] - \left(\frac{ik}{2\gamma}\right) \frac{1}{(v - z_0)^2} \tag{81}$$

so that our integral becomes

$$\begin{aligned} \sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^2 \cdot \eta_k)^{(s)}}{(v - z_0)^2 (v - z_0^*)} &= \left(\frac{ik}{2\gamma}\right)^2 \left[ \sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^2 \cdot \eta_k)^{(s)}}{v - z_0^*} - \sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^2 \cdot \eta_k)^{(s)}}{v - z_0} \right] \\ &\quad - \left(\frac{ik}{2\gamma}\right) \sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^2 \cdot \eta_k)^{(s)}}{(v - z_0)^2}. \end{aligned} \tag{82}$$

On the right hand side of this expression all the integrals are free of pinching singularities and therefore have finite limits as  $\gamma \rightarrow 0^+$ . The singularities of the original integral are now explicitly shown in the factors  $\gamma^{-2}$  and  $\gamma^{-1}$ .

The dispersion relation for the unstable mode  $\Lambda_k(z_0) = 0$  implies the identities

$$\int_{-\infty}^{\infty} dv \frac{\eta_k^{(e)}(v)}{v - z_0} = 1 + \int_{-\infty}^{\infty} dv \frac{\sum_s' \kappa^{(s)} \eta_k^{(s)}(v)}{v - z_0}, \tag{83}$$

$$\int_{-\infty}^{\infty} dv \frac{\eta_k^{(e)}(v)}{(v - z_0)^2} = -\Lambda_k'(z_0) - \int_{-\infty}^{\infty} dv \frac{\sum_s^* \kappa^{(s)} \eta_k^{(s)}(v)}{(v - z_0)^2}, \tag{84}$$

where we have used  $\kappa^{(e)} = -1$  and the primed sum indicates that the electron term is omitted. Substitution of (83) and (84) for the electron integrals in (82) yields

$$\sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^2 \cdot \eta_k)^{(s)}}{(v - z_0)^2 (v - z_0^*)} = \left(\frac{ik}{2\gamma}\right)^2 \left[ -2i \sum_s' \kappa^{(s)} (1 + \kappa^{(s)}) \text{Im} \int_{-\infty}^{\infty} \frac{dv \eta_k^{(s)}}{v - z_0} \right]$$

$$+ \left( \frac{ik}{2\gamma} \right) \left[ \Lambda'_k(z_0) + \sum_s ' \kappa^{(s)} (1 - \kappa^{(s)}) \int_{-\infty}^{\infty} \frac{dv \eta_k^{(s)}}{(v - z_0)^2} \right]. \quad (85)$$

Combining this with (80) we conclude that  $\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle$  typically diverges like  $\gamma^{-2}$  in the small growth rate limit. There are exceptions, however; for example, in the limit of fixed ions,  $\kappa^{(s)} \rightarrow 0$  for  $s \neq e$ , then we are left with

$$\sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^2 \cdot \eta_k)^{(s)}}{(v - z_0)^2 (v - z_0^*)} \rightarrow \left( \frac{ik}{2\gamma} \right) \Lambda'_k(z_0), \quad (86)$$

and the divergence is merely  $\gamma^{-1}$ .

We treat the remaining integral  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle$  in the same fashion. This integrand now has a fourth-order pole at  $v = z_0$  and a simple pole at  $v = z_0^*$ :

$$\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle = - \frac{2}{k^2 \Lambda'_k(z_0)} \sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^3 \cdot \eta_k)^{(s)}}{(v - z_0)^4 (v - z_0^*)}. \quad (87)$$

Again we make a partial fraction expansion

$$\begin{aligned} \frac{1}{(v - z_0)^4 (v - z_0^*)} &= \left( \frac{ik}{2\gamma} \right)^4 \left[ \frac{1}{v - z_0^*} - \frac{1}{v - z_0} \right] - \left( \frac{ik}{2\gamma} \right)^3 \frac{1}{(v - z_0)^2} - \left( \frac{ik}{2\gamma} \right)^2 \frac{1}{(v - z_0)^3} \\ &\quad - \left( \frac{ik}{2\gamma} \right) \frac{1}{(v - z_0)^4}, \end{aligned} \quad (88)$$

and eliminate the electron integrals to find

$$\begin{aligned} \sum_s \int_{-\infty}^{\infty} \frac{dv (\kappa^3 \cdot \eta_k)^{(s)}}{(v - z_0)^4 (v - z_0^*)} &= \left( \frac{ik}{2\gamma} \right)^4 \left[ + 2i \sum_s ' \kappa^{(s)} (1 - \kappa^{(s)2}) \text{Im} \int_{-\infty}^{\infty} \frac{dv \eta_k^{(s)}}{v - z_0} \right] \\ &\quad + \left( \frac{ik}{2\gamma} \right)^3 \left[ - \Lambda'_k(z_0) + \sum_s ' \kappa^{(s)} (1 - \kappa^{(s)2}) \int_{-\infty}^{\infty} \frac{dv \eta_k^{(s)}}{(v - z_0)^2} \right] \\ &\quad + \left( \frac{ik}{2\gamma} \right)^2 \left[ - \frac{\Lambda''_k(z_0)}{2} + \sum_s ' \kappa^{(s)} (1 - \kappa^{(s)2}) \int_{-\infty}^{\infty} \frac{dv \eta_k^{(s)}}{(v - z_0)^3} \right] \\ &\quad + \left( \frac{ik}{2\gamma} \right) \left[ - \frac{\Lambda'''_k(z_0)}{6} + \sum_s ' \kappa^{(s)} (1 - \kappa^{(s)2}) \int_{-\infty}^{\infty} \frac{dv \eta_k^{(s)}}{(v - z_0)^4} \right]. \end{aligned} \quad (89)$$

Thus for  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle$  the typical divergence is  $\gamma^{-4}$ .

Combining these results with  $p_1$  in (76), we obtain the asymptotic form of the cubic coefficient as  $\gamma \rightarrow 0^+$

$$p_1 = \frac{1}{\gamma^4} [c_1(\gamma) - \gamma d_1(\gamma) + \mathcal{O}(\gamma^2)], \quad (90)$$

where  $c_1$  and  $d_1$  are nonsingular functions of  $\gamma$  defined by

$$c_1(\gamma) = - \frac{k}{4\Lambda'_k(z_0)} \sum_s ' \kappa^{(s)} (1 - \kappa^{(s)2}) \text{Im} \left( \int_{-\infty}^{\infty} dv \frac{\eta_k^{(s)}}{v - z_0} \right), \quad (91)$$

$$d_1(\gamma) = \frac{1}{4} - \frac{1}{4\Lambda'_k(z_0)} \sum_s ' \kappa^{(s)} (1 - \kappa^{(s)2}) \int_{-\infty}^{\infty} dv \frac{\eta_k^{(s)}}{(v - z_0)^2}; \quad (92)$$

this result was briefly presented without derivation in an earlier paper.<sup>7</sup> When the leading term is nonzero at  $\gamma=0$  [i.e.,  $c_1(0) \neq 0$ ], then  $p_1$  diverges like  $\gamma^{-4}$  as a result of the divergence in  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle$ .

The significance of this result is best understood in light of our earlier discussion of scaling the amplitude  $A(t)$  to obtain an amplitude equation free of singular features at small growth rates. With our expression for  $p_1$ , the amplitude equation for the unstable mode is

$$\dot{A} = A \left[ \lambda + \frac{[c_1(0) + \mathcal{O}(\gamma)]}{\gamma^4} |A|^2 + p_2(\gamma) |A|^4 + \dots \right], \tag{93}$$

which we rewrite using polar variables  $A = \rho e^{-i\theta}$  as

$$\dot{\rho} = \gamma \rho + \frac{1}{\gamma^4} [\text{Re}(c_1(0)) + \mathcal{O}(\gamma)] \rho^3 + \text{Re}(p_2) \rho^5 + \dots, \tag{94}$$

$$\dot{\theta} = \omega - \frac{1}{\gamma^4} [\text{Im}(c_1(0)) + \mathcal{O}(\gamma)] \rho^2 - \text{Im}(p_2) \rho^4 + \dots. \tag{95}$$

Now we introduce a scaled amplitude  $\rho(t) = \gamma^\beta r(\gamma t)$  following (3) and anticipate that  $\delta=1$  is required to remove the singular effect of the linear term:

$$\frac{dr}{d\tau} = r + \frac{\gamma^{2\beta-1}}{\gamma^4} [\text{Re}(c_1(0)) + \mathcal{O}(\gamma)] r^3 + \gamma^{4\beta-1} \text{Re}(p_2) r^5 + \dots, \tag{96}$$

$$\dot{\theta} = \omega - \frac{\gamma^{2\beta}}{\gamma^4} [\text{Im}(c_1(0)) + \mathcal{O}(\gamma)] r^2 - \gamma^{4\beta} \text{Im}(p_2) r^4 + \dots. \tag{97}$$

The exponent  $\beta$  should be chosen so that the nonlinear coefficients have finite limits as  $\gamma \rightarrow 0^+$ . The most stringent requirements on  $\beta$  arise from the coefficients in the  $dr/d\tau$  equation, and the cubic term requires  $2\beta \geq 5$  to absorb the  $\gamma^{-4}$  singularity. Setting  $\beta=5/2$  suffices to cancel the singularities in  $p_1$  and allow the cubic nonlinearity to formally balance with the linear term at small  $\gamma$ :

$$\frac{dr}{d\tau} = r + [\text{Re}(c_1(0)) + \mathcal{O}(\gamma)] r^3 + \gamma^9 \text{Re}(p_2) r^5 + \dots, \tag{98}$$

$$\dot{\theta} = \omega - \gamma [\text{Im}(c_1(0)) + \mathcal{O}(\gamma)] r^2 - \gamma^{10} \text{Im}(p_2) r^4 + \dots. \tag{99}$$

This choice for  $\beta$  assumes that  $\text{Re}(c_1(0)) \neq 0$  and that the singularities at fifth (and higher) order are also removed by the same scaling that absorbs the cubic singularity. Typically the condition  $\text{Re}(c_1(0)) \neq 0$  is satisfied, but there are interesting exceptions and they are discussed next. The adequacy of the  $\beta=5/2$  scaling in controlling higher-order singularities is studied in Sec. IV.

**D. Special cases:  $c_1(0) = 0$**

Evaluating  $c_1(0)$  as  $\gamma \rightarrow 0^+$  yields

$$c_1(0) = - \frac{\pi k}{4 \Lambda'_k(v_p + i0)} \sum'_s \kappa^{(s)} (1 - \kappa^{(s)2}) \eta_k^{(s)}(v_p(0)), \tag{100}$$

where  $v_p(0)$  denotes the linear phase velocity at zero growth rate. Since the first factor ( $\pi k / 4 \Lambda'_k$ ) cannot vanish, the special case  $c_1(0) = 0$  will only arise if the species sum (which excludes the electrons) vanishes. By carefully adjusting the parameters of a given model, one can presumably arrange for exact cancellations between different terms in this sum, but there is little motivation to study such artificial situations.

However, there are three natural circumstances in which  $c_1(0) = 0$  because every term in the sum vanishes independently, and the cubic coefficient is clearly less singular:

- (a) Infinitely massive ions:  $\kappa^{(s)}=0$  for all  $s \neq e$ .
- (b) Zero slope for the resonant ions:  $\eta_k^{(s)}(v_p(0))=0$  for all  $s \neq e$ .
- (c) An electron-positron plasma:  $\kappa^{(p)2}=1$  for positrons.

In all three cases, the cubic coefficient has the asymptotic form

$$p_1 = -\frac{1}{\gamma^3} [d_1(0) + \mathcal{O}(\gamma)] \tag{101}$$

with  $d_1(0)$  given by the full formula (92) for the second case and  $d_1(0) = 1/4$  in the first and third cases.

In the rescaled equation (96), this weakened singularity requires only that  $2\beta \geq 4$ , and setting  $\beta=2$  yields

$$\frac{dr}{d\tau} = r + [\text{Re}(d_1(0)) + \mathcal{O}(\gamma)]r^3 + \gamma^7 \text{Re}(p_2)r^5 + \dots, \tag{102}$$

$$\dot{\theta} = \omega - [\text{Im}(d_1(0)) + \mathcal{O}(\gamma)]r^2 - \gamma^8 \text{Im}(p_2)r^4 + \dots. \tag{103}$$

In (I), we have shown that the exponent  $\beta=2$  suffices to absorb the singularities in (102) and (103) to all orders for a plasma with fixed ions; this exponent corresponds to the relatively familiar trapping scaling for the electric field of an unstable mode. In Sec. V, we provide a new proof of this result and also establish the same conclusion for the case (b) where each ion distribution is flat at the phase velocity:  $\eta_k^{(s)}(v_p)=0$  for all  $s \neq e$ . This will occur, for example, if the ions are sufficiently cold and do not populate the resonant region in velocity space. On the other hand, when there is an exact reflection symmetry  $F_0(v)=F_0(-v)$ , we can have an unstable mode with  $v_p=0$  and  $\eta_k^{(s)}(0)=0$  holds even though the ion distribution function is nonzero at  $v=0$ . This latter situation arises for instabilities of reflection-symmetric equilibria due to real eigenvalues, e.g., a reflection-symmetric two-stream instability.

Our understanding of the third example, an electron-positron plasma, is less complete. In Sec. V B we obtain by explicit calculation  $\text{Re}(p_2) \sim \gamma^{-8}$  so the rescaled fifth-order term in (102) remains singular unless  $\beta \geq 9/4$ . However, the singularity structure of the full expansion is more complicated and we have not determined the effects of singularities at higher order.

#### IV. SINGULARITY STRUCTURE OF THE EXPANSION

Our goal is a systematic analysis of the singularities of  $p_j$  to all orders in the amplitude equation (54). The detailed calculation of  $p_j$  rapidly becomes prohibitively laborious, but the recursion relations (57)–(59) determining the higher order coefficients in terms of lower order quantities (cf. Table I) can be analyzed to determine the properties of  $p_j$ . This study requires an accurate estimate of the pinching singularities in the integrals found in  $p_j$ . For this purpose we introduce an “index” which allows the divergence of a given integral to be assessed by a simple counting procedure. As in our discussion of the cubic coefficient, we assume the analytic singularities of  $F_0$  do not affect the divergence behavior; the precise assumption on  $F_0$  is given in (108) below.

##### A. Definition of the index

At every order in the expansion, the pinching singularities have a common structure. For  $n > 0$ , define

$$D_n(\alpha, v) \equiv \frac{1}{(v - \alpha_1)(v - \alpha_2) \cdots (v - \alpha_n)}, \tag{104}$$

where  $\alpha \equiv (\alpha_1, \dots, \alpha_n)$  and also define  $D_0(\alpha, v) \equiv 1$ . Evaluating  $p_j$  for  $j \geq 1$  involves integrands of the form

$$\mathcal{G}(v) = D_m(\beta, v) * D_n(\alpha, v) \sum_s (\kappa^{(s)})^m \frac{\partial^q \eta_l^{(s)}}{\partial v^q} \tag{105}$$

with  $m + n \geq 1$ ,  $m' \geq 1$  and  $q \geq 0$ . The poles in  $\mathcal{S}(v)$  may be written as

$$\alpha_j = z_0 + i\gamma\nu_j/k, \quad j = 1, \dots, n, \tag{106}$$

$$\beta_j^* = z_0^* - i\gamma\zeta_j/k, \quad j = 1, \dots, m; \tag{107}$$

since they always lie along the vertical line  $\text{Re}(v) = v_p(\gamma)$ . The specific pole locations are given by the numbers  $\nu_j \geq 0$  and  $\zeta_j \geq 0$  which vary depending on the integral. In all cases, these numbers are independent of  $F_0$ ; in particular  $\nu_j$  and  $\zeta_j$  are independent of  $\gamma$ .

The complex-analytic singularities of  $F_0$  are assumed to be unimportant for the amplitude expansions in the following sense. For any  $n \geq 1$  and  $q \geq 0$ , we assume that

$$\lim_{\gamma \rightarrow 0^+} \left| \int_{-\infty}^{\infty} dv D_n(\alpha, v) \frac{\partial^q \eta_l^{(s)}}{\partial v^q} \right| < \infty \tag{108}$$

holds for each species and arbitrary choices of  $\nu_j$  in (106). For example, it is sufficient for each distribution function  $F_0^{(s)}(v)$  to be analytic on a neighborhood of  $v_p(0)$ , the linear phase velocity at zero growth rate; in this case the limit in (108) can be evaluated by the Plemelj formulas for Cauchy integrals.<sup>14</sup>

The index of  $\mathcal{S}(v)$  in (105) is defined by

$$\text{Ind}[\mathcal{S}] \equiv m + n + q - 1, \tag{109}$$

and  $\text{Ind}[\mathcal{S}] \geq 0$  since we assume  $m + n \geq 1$ . For example, for  $\psi$  in (20), the integrand of  $\Gamma_{2,0}$  in (78) and the integrand of  $\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle$  in (80), we obtain the indices

$$\text{Ind} \left[ \sum_s \kappa^{(s)} \psi^{(s)} \right] = 0, \quad \text{Ind} \left[ \frac{\sum_s (\kappa^2 \cdot \eta_k)^{(s)}}{(v - z_0)^3} \right] = 2, \quad \text{Ind} \left[ \frac{\sum_s (\kappa^2 \cdot \eta_k)^{(s)}}{(v - z_0)^2 (v - z_0^*)} \right] = 2. \tag{110}$$

If  $mn = 0$ , our assumption (108) ensures that  $\int dv \mathcal{S}(v)$  is nonsingular, but when  $mn \neq 0$ , then as  $\gamma \rightarrow 0^+$ , the integral of  $\mathcal{S}$  develops a pinching singularity at the phase velocity  $v_p(0)$ . The index of  $\mathcal{S}$  estimates the worst possible divergence of  $\int dv \mathcal{S}$  due to this singularity.

*Lemma IV.1:* For  $\mathcal{S}(v)$  in (105) with  $mn \neq 0$ , the integral of  $\mathcal{S}$  satisfies

$$\lim_{\gamma \rightarrow 0^+} \gamma^J \left| \int_{-\infty}^{\infty} dv \mathcal{S}(v) \right| < \infty \tag{111}$$

with  $J = \text{Ind}[\mathcal{S}]$ . However, if

$$\sum_s' \kappa^{(s)} [(-1)^{m'} + (\kappa^{(s)})^{m'-1}] \eta_k^{(s)}(v_p(0)) = 0, \tag{112}$$

then the integral of  $\mathcal{S}$  is less singular and (111) holds with  $J = \text{Ind}[\mathcal{S}] - 1$ . In (112) the prime on the summation indicates that the sum omits the electron species.

*Proof:* By integrating by parts  $q$  times we can reduce to the case with  $q = 0$ . With  $q = 0$ , consider the simplest possibility  $m + n = 2$ , and make a partial fraction expansion of the integrand

$$\int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_1)(v - \beta_1^*)} = \frac{(-ik/\gamma)}{(2 + \nu_1 + \zeta_1)} \left[ \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_1)} - \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \beta_1^*)} \right]. \tag{113}$$

The integrals on the right are nonsingular with limits at  $\gamma = 0$  given by the Plemelj formula

$$\lim_{\gamma \rightarrow 0^+} \left[ \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_1)} - \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \beta_1^*)} \right] = 2\pi i \sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}(v_p(0)). \tag{114}$$

From the limit  $\gamma \rightarrow 0^+$  of the dispersion relation  $\Lambda_k(z_0) = 0$ , we obtain  $\sum_s (\kappa^{(s)}) \eta_k^{(s)}(v_p(0)) = 0$  for the imaginary part. Since  $\kappa^{(e)} = -1$ , this can be rewritten as  $\eta_k^{(e)} = \sum'_s (\kappa^{(s)}) \eta_k^{(s)}(v_p(0))$ , and used to eliminate  $\eta_k^{(e)}$  from (114):

$$\begin{aligned} &\lim_{\gamma \rightarrow 0^+} \left[ \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_1)} - \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \beta_1^*)} \right] \\ &= 2\pi i \left( \frac{k^2}{l^2} \right) \sum'_s \kappa^{(s)} [(-1)^{m'} + (\kappa^{(s)})^{m'-1}] \eta_k^{(s)}(v_p(0)). \end{aligned} \tag{115}$$

This proves the lemma for  $m+n=2$ . It is important that the limit in (115) is independent of the parameters  $\nu_1$  and  $\zeta_1$  that locate the poles  $\alpha_1$  and  $\beta_1$  for  $\gamma > 0$ .

For integrals with  $m+n > 2$ , by expanding the integrand in partial fractions, they can be reexpressed in terms of integrals with  $m+n-1$  multiplied by a factor of  $\gamma^{-1}$ . A simple induction argument then establishes (111) for general  $mn \neq 0$ . (Recall that if  $mn=0$  then the integrals are nonsingular.)

The general significance of the condition in (112) becomes clear from the form of the partial fraction expansion. For example, consider the expansion for  $(m,n) = (2,1)$ :

$$\begin{aligned} \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_1)(v - \alpha_2)(v - \beta_1^*)} &= \frac{(-ik/\gamma)}{(2 + \nu_1 + \zeta_1)} \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_1)(v - \alpha_2)} \\ &\quad - \frac{(-ik/\gamma)^2}{(2 + \nu_1 + \zeta_1)(2 + \nu_2 + \zeta_1)} \left[ \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_2)} \right. \\ &\quad \left. - \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \beta_1^*)} \right]. \end{aligned} \tag{116}$$

The second term is the dominant singularity  $\gamma^{-2}$  and at  $\gamma=0$  the bracketed integrals yield the same factor obtained previously in (115). When this factor is zero, the singularity of the  $(m,n) = (2,1)$  integral is reduced from 2 to 1.

In the partial fraction expansion of the general case, the most singular terms that diverge according to (111) will always be proportional to the difference of two nonsingular integrals:

$$\left[ \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \alpha_j)} - \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^{m'} \eta_l^{(s)}}{(v - \beta_{j'}^*)} \right] \tag{117}$$

whose limit at  $\gamma=0$  is given by (115). Thus when this limit is zero, the singularity of the term must drop by 1. This proves the second part of the lemma.  $\square$

The exceptional situation defined by the condition in (112) is a generalization of the feature noted previously in our analysis of the cubic coefficient. Setting  $m'=3$  in (112) yields  $c_1(0) = 0$  from (100); the special circumstance that corresponded to a less singular integral at third order.

Our application of Lemma IV.1 to the recursion relations for  $I_{m,j}$  requires generalization of the index to allow for sums of functions with well-defined indices and products of  $\mathcal{S}$  with singular functions of  $\gamma$ . In each case, the generalized index is defined so that Eq. (111) remains true, i.e., the index for the composite function gives the *maximal* possible divergence of its integral.

First, if  $\mathcal{S}_1(v)$  and  $\mathcal{S}_2(v)$  have indices satisfying  $\text{Ind}[\mathcal{S}_1] \geq \text{Ind}[\mathcal{S}_2]$  then we define the index of the sum to be the larger index:

$$\text{Ind}[\mathcal{S}_1 + \mathcal{S}_2] \equiv \text{Ind}[\mathcal{S}_1]. \tag{118}$$

Clearly,  $\text{Ind}[\mathcal{S}_1]$  gives the maximal possible divergence of  $\int dv(\mathcal{S}_1 + \mathcal{S}_2)$ . Second, if  $q(\gamma)$  is a function of  $\gamma$  with the asymptotic behavior  $q(\gamma) \sim \gamma^{-\nu}$  as  $\gamma \rightarrow 0^+$ , then we define the index of  $q(\gamma)\mathcal{S}(v)$  to be

$$\text{Ind}[q\mathcal{S}] \equiv \text{Ind}[\mathcal{S}] + \nu. \tag{119}$$

Estimates of the form (111) still hold for  $q\mathcal{S}$  with  $J = \text{Ind}[q\mathcal{S}]$ . This completes the definition of the index.

By applying (118) and (119), the indices of  $I_{m,j}$  and  $h_{m,j}$  may be determined. For example, from (66) and (69) we have the indices

$$\text{Ind}\left[\sum_s \kappa^{(s)} I_{0,0}^{(s)}\right] = 1, \quad \text{Ind}\left[\sum_s \kappa^{(s)} I_{2,0}^{(s)}\right] = 1, \tag{120}$$

and from (77) and (79)

$$\text{Ind}\left[\sum_s \kappa^{(s)} h_{0,0}^{(s)}\right] = 2, \quad \text{Ind}\left[\sum_s \kappa^{(s)} h_{2,0}^{(s)}\right] = 2. \tag{121}$$

The index for  $h_{2,0}^{(s)}$  is correct provided  $\Gamma_{2,0}$  does not diverge more strongly than  $\gamma^{-2}$ . We can confirm this immediately from the index in (110); in fact,  $\Gamma_{2,0}$  is nonsingular, as noted already in Sec. III C. In Sec. IV B, these indices are applied to determine the singularity of  $p_1$ . We stress again that when applied to a composite function  $\mathcal{S}(v)$  the estimate in (111) does not necessarily determine the true singularity, but only an upper bound on the possible divergence of the integral.

There are several immediate consequences of the index definition worth stating. First, complex conjugation does not alter the index,  $\text{Ind}[\mathcal{S}] = \text{Ind}[\mathcal{S}^*]$ ; second, if  $G(v)$  has a well-defined index, then dividing  $G$  by  $(v - \alpha)$  or  $(v - \beta^*)$  simply increases the index of  $G$  by one:

$$\text{Ind}[\mathcal{S}/(v - \alpha)] = \text{Ind}[\mathcal{S}/(v - \beta^*)] = \text{Ind}[\mathcal{S}] + 1. \tag{122}$$

Here  $\alpha, \beta$  are defined as in (106) and (107) and thereby increase the index of every term in  $G$  by one; this implies (122). Finally, from (105), differentiating  $\mathcal{S}$  also raises the index by one,  $\text{Ind}[\partial_v \mathcal{S}] = 1 + \text{Ind}[\mathcal{S}]$ .

### B. Analysis of $p_1$ using the index

The above relations along with recursion relations determine the indices of  $I_{m,j}$  and  $h_{m,j}$  to all orders. This information permits an estimate of the singularities of the coefficients  $p_j$  for  $j \geq 1$ . In practice, this must be done recursively following the organization of the amplitude expansions in Table I. The procedure is illustrated here for the first level of Table I; more general results on the index to all orders are provided in Secs. IV C and V.

Table I begins with  $\psi$ ,  $h_{0,0}$  and  $h_{2,0}$  whose indices were evaluated in (110) and (121). From this information, we can estimate the integrals required to evaluate  $p_1$  in (76). The integrand of  $\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle$  has index 2, and the integrands of  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle$  and  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0} \rangle$  have index 4. In each case, the factor  $\partial_v \tilde{\psi}^{(s)*}$  increases the index by two from the values obtained in (110) and (121). Now Lemma IV.1 tells us that  $\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle$  cannot diverge more strongly than  $\gamma^{-2}$ , and  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle$  and  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0} \rangle$  cannot diverge more strongly than  $\gamma^{-4}$ .

These conclusions imply that the singularity of  $p_1$  cannot be worse than  $\gamma^{-4}$ ; this was found in Sec. III C by an explicit calculation of  $p_1$ . The explicit evaluation, of course, provided the important additional result that the  $\gamma^{-4}$  singularity actually occurs unless one of the exceptional cases corresponding to  $c_1(0) = 0$  is considered. The foregoing index analysis is easier but yields less complete information.

### C. The main result

The main result on the singularity structure of the amplitude expansions can now be proved.

**Theorem IV.1:** For  $j \geq 1$ , the coefficients in the expansion of the amplitude equation (54) satisfy

$$\lim_{\gamma \rightarrow 0^+} \gamma^{5j-1} |p_j| < \infty. \tag{123}$$

Let  $J_{m,j} \equiv (2m + 5j - 3) + 4\delta_{m,0} + 5\delta_{m,1}$ , then for  $j \geq 0, m \geq 0$  and  $m' \geq 0$ , the indices of  $I_{m,j}^{(s)}$  and  $h_{m,j}^{(s)}$  obey

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} I_{m,j}^{(s)} \right] \leq J_{m,j}. \tag{124}$$

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} h_{m,j}^{(s)} \right] \leq J_{m,j} + 1, \tag{125}$$

and the integrals in (72) and (75) satisfy

$$\lim_{\gamma \rightarrow 0^+} \gamma^{J_{m,j}+1} |\Gamma_{m,j}| < \infty, \tag{126}$$

$$\lim_{\gamma \rightarrow 0^+} \gamma^{J_{m,j}+3-\delta_{m,1}} |\langle \partial_v \tilde{\psi}, \kappa \cdot h_{m,j} \rangle| < \infty. \tag{127}$$

In addition, integrals over  $I_{1,j}^{(s)}(v)$  satisfy

$$\lim_{\gamma \rightarrow 0^+} \gamma^{J_{1,j}+n-1} \left| \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} I_{1,j}^{(s)}(v) \right| < \infty \tag{128}$$

for  $n \geq 1$  where  $D_n(\alpha, v)$  is defined in (104) with upper half-plane poles  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  of the form (106) but otherwise arbitrary.

The relations in (124)–(128) are of interest chiefly for their utility in proving (123). We are able to prove (124) and (125) as upper bounds, but we expect them to hold as equalities in most cases. For  $m = 1$ , the bounds in (126) and (127) give stronger estimates on the integrals than one would obtain simply from the index of the integrand calculated using (124) and (125). Similarly, the bounds in (128) are needed because they provide sharper estimates for the  $m = 1$  case than are available using the index information (124) and (125) alone.

Even though  $h_{1,j}$  and  $h_{1,j}^*$  have the same index, it is very important to recognize that the conclusion in (127) may not hold for  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,j}^* \rangle$ . The most singular terms in  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,j} \rangle$  are not in general the most singular terms in  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,j}^* \rangle$ ; for  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,j}^* \rangle$ , if we wish to avoid explicitly evaluating the integral, our only means of estimating the divergence is by calculating the index of the integrand from (125).

*Proof:* (1) The proof is by induction using the organization of Table I. In Secs. IV A and IV B, the relations in (123)–(127) have been explicitly verified for  $I_{0,0}, I_{2,0}, h_{0,0}, h_{2,0}, \Gamma_{0,0}, \Gamma_{2,0}$ , and  $p_1$ . This proves the theorem for the first level of Table I, but does not establish the validity of (128) for the initial case  $j = 0$  which enters in the second level of Table I. By substituting the recursion relation (68) for  $I_{1,0}$  into (128) we obtain

$$\begin{aligned}
 & \gamma^{3+n} \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} I_{1,0}^{(s)}(v) \\
 &= \gamma^{3+n} \left(\frac{i}{k}\right) \int_{-\infty}^{\infty} dv D_n(\alpha, v) \partial_v \left( \kappa \cdot \left[ h_{0,0} - h_{2,0} + \frac{1}{2} \psi^* \Gamma_{2,0} \right] \right) \\
 & \quad + \gamma^{3+n} \left(\frac{i}{k}\right) \left\langle \partial_v \tilde{\psi}, \kappa \cdot \left[ h_{0,0} - h_{2,0} + \frac{1}{2} \psi^* \Gamma_{2,0} \right] \right\rangle \int_{-\infty}^{\infty} dv D_n(\alpha, v) \psi(v). \tag{129}
 \end{aligned}$$

On the right hand side, the index of the first integrand is  $(n + 3)$  so the factor of  $\gamma^{n+3}$  ensures a finite limit. In the second term, the integral over  $D_n \psi$  is nonsingular since all poles are in the upper half-plane, but the coefficient  $\langle \partial_v \tilde{\psi}, \kappa \cdot [h_{0,0} - h_{2,0} + \frac{1}{2} \psi^* \Gamma_{2,0}] \rangle$  diverges like  $\gamma^{-4}$  (cf. Sec. III C). Thus for  $n \geq 1$  the second term will also have a finite limit; this verifies (128) for  $I_{1,0}$ .

(2) The relations (123)–(128) are extended by induction to all coefficients  $I_{m,j}$ ,  $h_{m,j}$ , and  $p_j$  using the recursion relations (57)–(59) and (66)–(71). Assume that (123)–(128) are valid down to some arbitrary level of Table I, and consider what the recursion relations imply for the coefficients evaluated at the next level. We use  $(m_1, j_1)$  to denote subscripts of coefficients such as  $I_{m_1, j_1}$ ,  $h_{m_1, j_1}$  and  $p_{j_1}$  that are to be evaluated from lower order quantities assumed to satisfy (123)–(128).

The induction argument has three parts. First, we show that the properties (125)–(127) must hold for  $h_{m_1, j_1}$  and  $\Gamma_{m_1, j_1}$  if  $I_{m_1, j_1}$  satisfies two relations, namely the index identity (124)

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} I_{m_1, j_1}^{(s)} \right] \leq J_{m_1, j_1}, \tag{130}$$

and the  $m_1 = 1$  estimate (128)

$$\lim_{\gamma \rightarrow 0^+} \gamma^{J_{m_1, j_1} + n - 1} \left| \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} I_{1, j_1}^{(s)}(v) \right| < \infty. \tag{131}$$

In the second part of the proof, we verify that (130) and (131) do hold for  $I_{m_1, j_1}$ . Finally, in the third part of the proof, we verify that (125)–(127) imply (123) for  $p_{j_1}$ . It is important to note from Table I that the quantities  $h_{m_1, j_1}$  depend on  $p_j$  for  $j < j_1$ , but are independent of  $p_{j_1}$  so the reasoning is not circular.

(3) **(Part 1):** Suppose that (130) and (131) hold, then the three relations (125)–(127) are easily obtained as follows. First, consider (126) which holds trivially for  $m_1 = 0$  since  $\Gamma_{0, j_1} = 0$ . For  $m_1 \geq 2$ , (126) follows from (130) and the identity (72). For  $m_1 = 1$ , we have to allow for the fact that  $\Lambda_k(z_{1, j_1}) \sim \gamma$  as  $\gamma \rightarrow 0^+$ , and apply (131) to estimate the integral in (72). Next (125) follows by applying (130) and (126) to the identity (73). Each term in (73) separately has a maximum index of  $(J_{m_1, j_1} + 1)$  so this must bound the index of the sum.

Finally consider (127). For  $m_1 \neq 1$ , this relation follows from (125), and for  $m_1 = 1$  we evaluate (127) using the identity in (73):

$$\begin{aligned}
 \gamma^{J_{1, j_1} + 2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{1, j_1} \rangle &= -\gamma^{5j_1 + 6} \left(\frac{i}{k}\right) \langle \partial_v \tilde{\psi}, \kappa \cdot I_{1, j_1} / (v - z_{1, j_1}) \rangle \\
 &\quad - \gamma^{5j_1 + 6} \Gamma_{1, j_1} \langle \partial_v \tilde{\psi}, \kappa^2 \cdot \eta_k / (v - z_{1, j_1}) \rangle. \tag{132}
 \end{aligned}$$

The first term on the right is nonsingular by virtue of (131), and the second term is also nonsingular by virtue of (126).

(4) **(Part 2):** Thus the crux of the matter is to verify (130) and (131) from the recursion relations for  $I_{m_1, j_1}^{(s)}$ , i.e., from (66)–(71). In this step, we assume (123)–(128) apply to the quantities appearing on the right hand side of these recursion relations, then the properties in (130) and (131) are verified term by term.

Verification of (130)—the index of  $I_{m_1, j_1}$ : The index of each term on the right hand side of the recursion relations for  $I_{m_1, j_1}^{(s)}(v)$  can be evaluated by applying (123)–(127); this exercise shows that all of these terms have an index less than or equal to  $J_{m_1, j_1}$  which establishes (130) for  $I_{m_1, j_1}^{(s)}$ . A few examples from (68), the recursion relation for  $(m_1, j_1) = (1, j_1)$ , illustrate these index calculations; note that in these examples  $J_{1, j_1} = 5j_1 + 4$ .

(a) The terms in (68) that depend on  $p_j$  have the form  $[(2+n)p_{j_1-n} + (1+n)p_{j_1-n}^*]h_{1,n}^{(s)}$ ; from (125) we have

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} h_{1,n}^{(s)} \right] \leq J_{1,n} + 1 = 5n + 5, \tag{133}$$

and from (123) the singularity of  $p_{j_1-n}$  is determined, hence

$$\text{Ind} \left[ p_{j_1-n} \sum_s (\kappa^{(s)})^{m'} h_{1,n}^{(s)} \right] \leq 5(j_1 - n) - 1 + (5n + 5) = 5j_1 + 4 \tag{134}$$

which is consistent with (130) for  $(m_1, j_1) = (1, j_1)$ .

(b) Next consider the term  $\mathcal{P}_\perp \kappa \cdot \partial_v h_{0, j_1}$  in (68); from the definition (49) we have that the index of  $\mathcal{P}_\perp \kappa \cdot \partial_v h_{0, j_1}$  is equal to the largest index obtained from the two terms  $\kappa \cdot \partial_v h_{0, j_1}$  and  $\psi(v) \langle \partial_v \tilde{\psi}, \kappa \cdot h_{0, j_1} \rangle$ . These terms have individual indices

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} \partial_v h_{0, j_1}^{(s)} \right] \leq 5j_1 + 3, \tag{135}$$

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} \psi^{(s)}(v) \langle \partial_v \tilde{\psi}, \kappa \cdot h_{0, j_1} \rangle \right] \leq 5j_1 + 4, \tag{136}$$

where the first index (135) comes from (125) and the second follows from (127). Hence

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} (\mathcal{P}_\perp \kappa \cdot \partial_v h_{0, j_1})^{(s)} \right] \leq 5j_1 + 4 \tag{137}$$

which is consistent with (130) for  $(m_1, j_1) = (1, j_1)$ .

(c) A more subtle example is the term  $\mathcal{P}_\perp \kappa \cdot \partial_v h_{1,n}^* \Gamma_{2, j_1 - n - 1}$  in (68) which expands to

$$\mathcal{P}_\perp \kappa \cdot \partial_v h_{1,n}^* \Gamma_{2, j_1 - n - 1} = \kappa \cdot \partial_v h_{1,n}^* \Gamma_{2, j_1 - n - 1} + \Gamma_{2, j_1 - n - 1} \psi(v) \langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,n}^* \rangle. \tag{138}$$

Now the index of each term can be evaluated to give

$$\text{Ind} \left[ \Gamma_{2, j_1 - n - 1} \sum_s (\kappa^{(s)})^{m'} \partial_v h_{1,n}^{(s)*} \right] \leq 5j_1 + 2, \tag{139}$$

$$\text{Ind} \left[ \Gamma_{2, j_1 - n - 1} \sum_s (\kappa^{(s)})^{m'} \psi^{(s)} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,n}^* \rangle \right] \leq 5j_1 + 4; \tag{140}$$

thus

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} (\mathcal{P}_\perp \kappa \cdot \partial_v h_{1,n}^*)^{(s)} \Gamma_{2, j_1 - n - 1} \right] \leq 5j_1 + 4 \tag{141}$$

which is consistent with (130) for  $(m_1, j_1) = (1, j_1)$ . In this last example, (139) is obtained from (125) and (126); the second result (140) differs from (136) since (127) cannot be applied and the

singularity of  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,n}^* \rangle$  must be estimated from the index in (125). In a similar way the index of each term appearing on the right in the recursion relations (66)–(71) for  $I_{m_1, j_1}(v)$  can be estimated with the final result in (130).

(5) *Verification of (131)*: We integrate the recursion relation for  $I_{1, j_1}(v)$  and examine the resulting expression to establish (131). When (68), the recursion for  $I_{1, j_1}$ , is inserted into (131) we obtain

$$\begin{aligned} & \gamma^{j_{1, j_1} + n - 1} \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} I_{1, j_1}^{(s)}(v) \\ &= -\gamma^{5j_1 + n + 3} \sum_{l=0}^{j_1 - 1} [(2+l)p_{j_1 - l} + (1+l)p_{j_1 - l}^*] \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} h_{1, l}^{(s)}(v) \\ &+ \gamma^{5j_1 + n + 3} \left(\frac{i}{k}\right) \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m' + 1} G^{(s)}(v) \\ &+ \gamma^{5j_1 + n + 3} \left(\frac{i}{k}\right) \langle \partial_v \tilde{\psi}, \kappa \cdot G \rangle \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} \psi^{(s)}(v), \end{aligned} \tag{142}$$

where  $G(v) = h_{0, j_1} - h_{2, j_1} + \dots$  denotes the collection of terms enclosed by brackets in (68). The relations (124)–(127) hold for these terms by assumption, and it is straightforward to verify that  $\text{Ind}[G] \leq 5j_1 + 2$ ; this immediately implies that the terms in (142) involving  $G$  are finite as  $\gamma \rightarrow 0^+$ . For the remaining terms in (142) we have  $p_{j_1 - l} \sim \gamma^{-5(j_1 - l) + 1}$  from (123), hence we need to prove

$$\lim_{\gamma \rightarrow 0^+} \gamma^{5l + 4 + n} \left| \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} h_{1, l}^{(s)}(v) \right| < \infty \quad (n \geq 1) \tag{143}$$

to establish (131). The formula in (125) bounds the index of the integrand in (143) by only  $(5l + n + 5)$  which is not sufficient. Instead, we evaluate (143) using the identity (73)

$$\begin{aligned} \int_{-\infty}^{\infty} dv D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} h_{1, l}^{(s)}(v) &= -\left(\frac{i}{k}\right) \int_{-\infty}^{\infty} dv D_n(\alpha, v) \frac{\sum_s (\kappa^{(s)})^{m'} I_{1, l}^{(s)}(v)}{(v - z_{1, l})} \\ &- \Gamma_{1, l} \int_{-\infty}^{\infty} dv D_n(\alpha, v) \frac{\sum_s (\kappa^{(s)})^{m' + 1} \eta_k^{(s)}}{(v - z_{1, l})}. \end{aligned} \tag{144}$$

The integral in the second term is nonsingular and  $\Gamma_{1, l} \sim \gamma^{-5l - 5}$  from (126), thus the desired estimate in (143) holds for the second term. It remains to verify

$$\lim_{\gamma \rightarrow 0^+} \gamma^{5l + 4 + n} \left| \int_{-\infty}^{\infty} dv D_n(\alpha, v) \frac{\sum_s (\kappa^{(s)})^{m'} I_{1, l}^{(s)}(v)}{(v - z_{1, l})} \right| < \infty \quad (n \geq 1) \tag{145}$$

for the first term in (144); this follows from (128) with  $j \rightarrow l$  and  $D_n \rightarrow D_n / (v - z_{1, l})$ .

The proof of (143) establishes (131) for  $I_{1, j_1}$ .

(6) **(Part 3): Verification of (123) for  $p_{j_1}$** : It is straightforward to check that application of (125)–(127) to the right hand side of the recursion relations (57)–(59) yields the estimate in (123). A few examples from (58) and (59) suffice to illustrate how this is carried out.

(a) Consider the term  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0, j_1 - 1} \rangle$  in (58); from (127) the singularity of this integral is at most

$$|\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0, j_1 - 1} \rangle| \sim \left(\frac{1}{\gamma}\right)^{3 + J_{0, j_1 - 1}} = \left(\frac{1}{\gamma}\right)^{5j_1 - 1} \tag{146}$$

which is consistent with (123).

(b) Other terms will involve products of integrals; for example,  $\Gamma_{2,j_1-1} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle$  in (58). Using (82) and (126) we determine that the singularity of this product is at most

$$|\Gamma_{2,j_1-1} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle| \sim \left(\frac{1}{\gamma}\right)^{1+J_{2,j_1-1}} \left(\frac{1}{\gamma}\right)^2 = \left(\frac{1}{\gamma}\right)^{5j_1-1} \tag{147}$$

which again is consistent with (123).

(c) A final example is the product  $\Gamma_{2,l} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,j_1-l-2}^* \rangle$  with  $0 \leq l \leq j_1 - 2$  from (59); applying (125) and (126) to this shows a maximum singularity of

$$|\Gamma_{2,l} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,j_1-l-2}^* \rangle| \sim \left(\frac{1}{\gamma}\right)^{1+J_{2,l}} \left(\frac{1}{\gamma}\right)^{3+J_{1,j_1-l-2}} = \left(\frac{1}{\gamma}\right)^{5j_1-1} \tag{148}$$

which is consistent with (123).

This completes the proof. □

### D. Implications of the main result

Theorem IV.1 allows us to evaluate the  $\gamma \rightarrow 0^+$  limit of the theory. Rewriting the amplitude equation (54) using polar variables  $A(t) = \rho(t) \exp(-i\theta(t))$ , and the scaling

$$\rho(t) = \gamma^{5/2} r(\gamma t) \tag{149}$$

from Sec. III C, yields

$$\frac{dr}{d\tau} = r + \sum_{j=1}^{\infty} [\gamma^{5j-1} \text{Re}(p_j)] r^{2j+1}, \tag{150}$$

$$\frac{d\theta}{dt} = \omega - \gamma \sum_{j=1}^{\infty} [\gamma^{5j-1} \text{Im}(p_j)] r^{2j}, \tag{151}$$

where  $\tau = \gamma t$ . The result (123) on the divergence of the coefficients  $p_j$  ensures that the terms in (150) and (151) are now finite as  $\gamma \rightarrow 0^+$ . Introducing the asymptotic coefficients

$$c_j(0) = \lim_{\gamma \rightarrow 0^+} [\gamma^{5j-1} p_j] \quad (j \geq 1) \tag{152}$$

the amplitude equation for a weakly unstable wave becomes

$$\frac{dr}{d\tau} = r + \sum_{j=1}^{\infty} [\text{Re}(c_j(0)) + \mathcal{O}(\gamma)] r^{2j+1}. \tag{153}$$

Unless  $\text{Re}(c_j(0)) = 0$  in each term, the scaling in (149) is required to obtain a finite theory. The circumstances that could force  $c_1(0) = 0$  were discussed in Sec. III C, and will be analyzed more completely in Sec. V.

The electric field is obtained from Poisson's equation  $(imk)E_{mk}(t) = \sum_s \int_{-\infty}^{\infty} dv f_{mk}^u(s)$  using the form of  $f_{mk}^u$  in (44)

$$E_{mk}(t) = \begin{cases} -iA(1 + \sigma\Gamma_1(\sigma))/k, & m = 1 \\ -iA^m\Gamma_m(\sigma)/mk, & m > 1. \end{cases} \tag{154}$$

The small growth rate behavior of  $\Gamma_m(\sigma)$  can be inferred from (126) and the expansion  $\Gamma_m(\sigma) = \sum_j \Gamma_{m,j} \sigma^j$ . Using  $\sigma^j = \gamma^{5j} r^{2j}$ , we find an expansion of  $\Gamma_m$  with a nonsingular limit

$$\gamma^{2m-2+5\delta_{m,1}} \Gamma_m = \sum_{j=0}^{\infty} \gamma^{j(m,j)+1} \Gamma_{m,j} r^{2j}, \tag{155}$$

and this motivates our definition of  $\Gamma_m^c$

$$\Gamma_m^c = \lim_{\gamma \rightarrow 0^+} \gamma^{2m-2+5\delta_{m,1}} \Gamma_m = \sum_{j=0}^{\infty} b_{m,j}(0) r^{2j}, \tag{156}$$

where

$$b_{m,j}(0) \equiv \lim_{\gamma \rightarrow 0^+} [\gamma^{j m_j + 1} \Gamma_{m,j}]. \tag{157}$$

With this notation, the asymptotic form of the Fourier expansion  $E(x,t) = \sum_m E_{mk}(t) \exp(imkx)$  of the field can be evaluated

$$E(x,t) = -\frac{i\gamma^{5/2}}{k} \left[ r(\tau) (1 + \Gamma_1^c r(\tau)^2 + \mathcal{O}(\gamma)) e^{i(kx - \theta(t))} + \sum_{m=2}^{\infty} \gamma^{(m-1)/2} \frac{(\Gamma_m^c r(\tau)^m + \mathcal{O}(\gamma))}{m} e^{im(kx - \theta(t))} \right] + \text{c.c.} \tag{158}$$

The asymptotic form may be rewritten more simply as

$$\frac{iE(x,t)}{\gamma^{5/2}} = \left[ \frac{r(\tau)}{k} (1 + \Gamma_1^c r(\tau)^2) e^{i(kx - \theta(t))} + \text{c.c.} \right] + \mathcal{O}(\sqrt{\gamma}); \tag{159}$$

this describes an electric field comprised of a single Fourier component whose coefficient scales like  $\gamma^{5/2}$  and evolves nonlinearly according to the asymptotic amplitude equation (153). The  $\gamma^{5/2}$  scaling yields a dramatically smaller field than would be expected from the trapping scaling  $\gamma^2$  characteristic of a model with infinitely massive ions.

An analogous discussion of the asymptotic distribution function  $f(x,v,t)$  is substantially more complicated because of singularities in  $v$  that emerge as  $\gamma \rightarrow 0^+$ . This issue is briefly treated in Sec. VI, but we defer a full analysis to a companion paper.<sup>15</sup>

### V. SPECIAL CASES

Section III D identified three settings for which  $c_1(0) = 0$  and the  $\gamma^{-4}$  singularity of  $p_1$  is replaced by a  $\gamma^{-3}$  divergence:

- (a) Infinitely massive ions:  $\kappa^{(s)} = 0$  for all  $s \neq e$ ,
- (b) zero slope for the resonant ions:  $\eta_k^{(s)}(v_p(0)) = 0$  for all  $s \neq e$ ,
- (c) an electron-positron plasma:  $\kappa^{(p)^2} = 1$  for positrons ( $s = p$ ).

From the partial fraction expansion in Lemma IV.1, we found the general condition required to reduce the singularity of a standard integral,

$$\int_{-\infty}^{\infty} dv \mathcal{F}(v) = \int_{-\infty}^{\infty} dv D_m(\beta, v) * D_n(\alpha, v) \sum_s (\kappa^{(s)})^{m'} \frac{\partial^q \eta_l^{(s)}}{\partial v^q} \quad (mn > 0) \tag{160}$$

from  $\gamma^{-m-n-q+1}$  to  $\gamma^{-m-n-q+2}$ ; namely,

$$\sum'_s \kappa^{(s)} [(-1)^{m'} + (\kappa^{(s)})^{m'-1}] \eta_k^{(s)}(v_p(0)) = 0, \tag{161}$$

where the primed sum omits the electrons. The specific instance of (161) that yields  $c_1(0) = 0$  corresponds to  $m' = 3$ .

The validity of this general condition varies between the three settings listed above. For both (a) and (b), (161) holds quite generally for all integers  $m' \geq 0$ ; however, for the third setting (c), the condition reduces to

$$[(-1)^{m'} + 1] \eta_k^{(p)}(v_p(0)) = 0, \tag{162}$$

and this will be generally satisfied only for  $m'$  odd.

**A. Cases (a) and (b)**

It is relatively simple to adapt Theorem IV.1 to accommodate the first two settings where (161) holds uniformly. The following result restates Theorem IV.1 of (I) under more general assumptions [note that the definition of the index in this paper differs from (I)].<sup>1</sup>

**Theorem V.1:** Assume that (161) holds for all  $m' \geq 0$ ; then for  $j \geq 1$ , the coefficients in the expansion of the amplitude equation (54) satisfy

$$\lim_{\gamma \rightarrow 0^+} \gamma^{4j-1} |p_j| < \infty. \tag{163}$$

Let  $J_{m,j} \equiv (2m + 5j - 3) + 4\delta_{m,0} + 5\delta_{m,1}$ , then for  $j \geq 0$ ,  $m \geq 0$  and  $m' \geq 0$ , the indices of  $I_{m,j}^{(s)}$  and  $h_{m,j}^{(s)}$  obey

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} I_{m,j}^{(s)} \right] \leq J_{m,j} - j - \delta_{m,1}. \tag{164}$$

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} h_{m,j}^{(s)} \right] \leq J_{m,j} + 1 - j - \delta_{m,1}, \tag{165}$$

and the integrals in (72) and (75) satisfy

$$\lim_{\gamma \rightarrow 0^+} \gamma^{J_{m,j}-j} |\Gamma_{m,j}| < \infty, \tag{166}$$

$$\lim_{\gamma \rightarrow 0^+} \gamma^{J_{m,j}+2-j-\delta_{m,1}} |\langle \partial_v \tilde{\psi}, \kappa \cdot h_{m,j} \rangle| < \infty. \tag{167}$$

The proof is simpler since (165)–(167) follow directly from (164) and there is no need for special estimates such as (128) in the  $m = 1$  case.

*Proof:* (1) The induction proof follows that for Theorem IV.1, and we only highlight the new features brought in by (161). The recursion relations for  $I_{m,j}$  are finite sums; each term of which has the form  $q(\gamma)\mathcal{S}(v)$  where  $\text{Ind}[\mathcal{S}(v)] \geq 1$  and the coefficient  $q(\gamma)$  may be singular as  $\gamma \rightarrow 0^+$ . According to Lemma IV.1, with (161) in force, if  $J = \text{Ind}[q(\gamma)\mathcal{S}(v)]$ , then  $\gamma^{J-1}q(\gamma)\int dv \mathcal{S}(v)$  remains finite as  $\gamma \rightarrow 0^+$ .

(2) We have already shown that (161) implies (163) for  $j = 1$  so the theorem has been proved for the first level of Table I. As before, we assume that (163)–(167) are valid down to some arbitrary level of Table I, and consider what the recursion relations imply for the coefficients evaluated at the next level. Again, we use  $(m_1, j_1)$  to denote subscripts of coefficients such as  $I_{m_1, j_1}^{(s)}$ ,  $h_{m_1, j_1}^{(s)}$  and  $p_{j_1}$  that are to be evaluated from lower order quantities assumed to satisfy (163)–(167). Following the organization of the previous proof, we first check that (164) for  $I_{m_1, j_1}^{(s)}$  implies (165)–(167) for  $h_{m_1, j_1}^{(s)}$ ,  $\Gamma_{m_1, j_1}$ , and  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{m_1, j_1} \rangle$ , respectively. Then (164) is verified from the recursion relation for  $I_{m_1, j_1}^{(s)}$ , and finally (163) is shown to follow from (165)–(167).

(3) **(Part 1):** Assume that (164) holds for  $I_{m_1, j_1}^{(s)}$ :

$$\text{Ind} \left[ \sum_s (\kappa^{(s)})^{m'} I_{m_1, j_1}^{(s)} \right] \leq J_{m_1, j_1} - j_1 - \delta_{m_1, 1}, \tag{168}$$

then the properties (165)–(167) are easily obtained as follows. First, the integrand of  $\Gamma_{m_1, j_1}$  in (72) has index

$$\text{Ind} \left[ \sum_s I_{m_1, j_1}^{(s)} / (v - z_{m,j}) \right] \leq J_{m_1, j_1} - j_1 - \delta_{m_1, 1} + 1 \tag{169}$$

from (164) so the expression

$$\gamma^{J_{m_1, j_1} - j_1 - \delta_{m_1, 1}} \int_{-\infty}^{\infty} dv \frac{\sum_s I_{m, j}^{(s)}(v)}{v - z_{m, j}} \tag{170}$$

remains finite as  $\gamma \rightarrow 0^+$ ; this proves (166) since the  $\delta_{m_1, 1}$  in the exponent compensates for the added singularity of  $\Lambda_k(z_{1, j_1})$  in (72). Next, (165) follows by applying (168) and (166) to the identity (73). Finally, the integrand in (167) has index

$$\text{Ind} \left[ \sum_s \partial_v \tilde{\psi}^{(s)} \kappa^{(s)} h_{m, j}^{(s)} \right] \leq J_{m_1, j_1} - j_1 - \delta_{m_1, 1} + 3 \tag{171}$$

from (165) so the bound in (167) follows from (161) and Lemma IV.1.

(4) **(Part 2): Verification of (168)—the index of  $I_{m_1, j_1}$ :** The consequences of (161) for the index calculation are incorporated in (163)–(166). The index of each term on the right hand side of the recursion relations for  $I_{m_1, j_1}^{(s)}(v)$  can be evaluated by applying these conclusions regarding the lower order coefficients; this is done exactly as in the proof of (130) in Theorem IV.1. The bound in (168) follows as before.

(5) **(Part 3): Verification of (163) for  $p_{j_1}$ :** It is straightforward to check that application of (165)–(167) to the right hand side of the recursion relations (57)–(59) yields the estimate in (163) just as in the proof of Theorem IV.1.  $\square$

In light of the discussion in Sec. IV D, the theorem tells us that the scaling in (149) is replaced by

$$\rho(t) = \gamma^2 r(\gamma t) \tag{172}$$

for the special cases (a) and (b). This modification in turn implies an electric field characterized by the ‘trapping scaling’  $E_k \sim \gamma^2$ ; the discussion in (I) provides detailed expressions for the asymptotic form of  $E(x, t)$  and  $f^{(e)}(x, v, t)$  in the limit  $\gamma \rightarrow 0^+$  when the ions are fixed.<sup>1</sup>

### B. An electron-positron plasma

The result  $p_j \sim \gamma^{-4j+1}$  for the two special cases just considered is not correct for the electron-positron system with  $\kappa^{(p)} = -\kappa^{(e)} = 1$ . Although  $p_1 \sim \gamma^{-3}$  does hold, we find at fifth order  $p_2 \sim \gamma^{-8}$  instead of  $p_2 \sim \gamma^{-7}$ . Thus the trapping scaling of the amplitude in (172) is not sufficient to obtain a finite expansion. We summarize the evaluation of  $p_2$  from (57)–(59), but leave the comprehensive analysis of this case as an open problem.

The exact recursion relation for  $p_2$  is

$$-ikp_2 = -\langle \partial_v \tilde{\psi}, \kappa \cdot (h_{0,1} - h_{2,1}) \rangle - \frac{1}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle \Gamma_{2,1} - \langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle \Gamma_{1,0} + \langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0} \rangle \Gamma_{1,0}^* - \left\{ \frac{1}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{1,0}^* \rangle \Gamma_{2,0} + \frac{\Gamma_{3,0}}{3} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0}^* \rangle - \frac{\Gamma_{2,0}^*}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot h_{3,0} \rangle \right\}, \tag{173}$$

and, in the bracketed terms,  $\Gamma_{2,0}$  and  $\Gamma_{3,0}$  are nonsingular so Theorem IV.1 ensures the divergence of these terms cannot exceed  $\gamma^{-4}$ ; hence they are subdominant:

$$-ikp_2 = -\langle \partial_v \tilde{\psi}, \kappa \cdot (h_{0,1} - h_{2,1}) \rangle - \frac{1}{2} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle \Gamma_{2,1} - \langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle \Gamma_{1,0} + \langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0} \rangle \Gamma_{1,0}^* + \mathcal{O}(\gamma^{-4}). \tag{174}$$

Here consider the two terms involving  $\Gamma_{1,0}$ . In the general case, Theorem IV.1 gives  $\Gamma_{1,0} \sim \gamma^{-5}$ , but this drops to  $\gamma^{-4}$  when we take (161) into account for  $m'$  odd. The integral  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0} \rangle$  is nonsingular so  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{2,0} \rangle \Gamma_{1,0}^* \sim \gamma^{-4}$  and we neglect this term also. Next  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle$  is written out to find

$$\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle = \frac{i}{2k\gamma} \langle \partial_v \tilde{\psi}, \kappa^2 \cdot \partial_v (\psi^* - \psi) \rangle; \tag{175}$$

this integrand has index equal to 3, but contains an odd power of  $\kappa$  so (161) holds and the divergence cannot exceed  $\gamma^{-2}$ . Overall,  $\langle \partial_v \tilde{\psi}, \kappa \cdot h_{0,0} \rangle \Gamma_{1,0}$  has a maximum singularity of  $\gamma^{-7}$ . A careful examination of the first term in (174) yields the same estimate  $\langle \partial_v \tilde{\psi}, \kappa \cdot (h_{0,1} - h_{2,1}) \rangle \sim \gamma^{-7}$ .

The dominant singularity in  $p_2$  arises from the remaining term

$$p_2 = -\frac{i\Gamma_{2,1}}{2k} \langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle + \mathcal{O}(\gamma^{-7}), \tag{176}$$

and from (85) and (115), we obtain

$$\langle \partial_v \tilde{\psi}, \kappa \cdot \psi^* \rangle = -\left(\frac{k}{\gamma}\right)^2 \left[ \frac{i\pi \eta_k^{(p)}(v_p(0))}{\Lambda_k'(v_p + i0)} + \mathcal{O}(\gamma) \right]. \tag{177}$$

The evaluation of  $\Gamma_{2,1}$  starts from (72)

$$\Gamma_{2,1} = \frac{-i/2k}{\Lambda_{2k}(z_{2,1})} \sum_s \int_{-\infty}^{\infty} dv \frac{I_{2,1}^{(s)}(v)}{v - z_{2,1}}, \tag{178}$$

and the recursion relation (70) for  $I_{2,1}$

$$I_{2,1}(v) = -2p_1 h_{2,0} + \frac{i}{k} \frac{\partial}{\partial v} \kappa \cdot \left\{ h_{1,0} + \psi \Gamma_{1,0} - h_{3,0} + \frac{1}{3} \psi^* \Gamma_{3,0} + \frac{1}{2} h_{0,0} \Gamma_{2,0} \right\}. \tag{179}$$

Note that  $\Lambda_{2k}(z_{2,1}) = 3/4 + \mathcal{O}(\gamma)$  from (21) so any singularities arise from the integral

$$\int_{-\infty}^{\infty} dv \frac{\sum_s I_{2,1}^{(s)}(v)}{v - z_{2,1}} = -2p_1 \int_{-\infty}^{\infty} dv \frac{\sum_s h_{2,0}^{(s)}(v)}{v - z_{2,1}} + \frac{i}{k} \int_{-\infty}^{\infty} \frac{dv}{(v - z_{2,1})^2} \sum_s \kappa^{(s)} \left[ h_{1,0}^{(s)} + \psi^{(s)} \Gamma_{1,0} - h_{3,0}^{(s)} + \frac{1}{3} \psi^{(s)*} \Gamma_{3,0} + \frac{1}{2} h_{0,0}^{(s)} \Gamma_{2,0} \right]. \tag{180}$$

The strongest singularity here occurs in the integral over  $h_{1,0}^{(s)}$ , and all other terms in (180) are subdominant; thus we can focus attention on the expression

$$\int_{-\infty}^{\infty} dv \frac{\sum_s I_{2,1}^{(s)}(v)}{v - z_{2,1}} = \frac{i}{k} \int_{-\infty}^{\infty} dv \frac{\sum_s \kappa^{(s)} h_{1,0}^{(s)}}{(v - z_{2,1})^2} + \mathcal{O}(\gamma^{-4}). \tag{181}$$

When  $h_{1,0}^{(s)}$  is evaluated using (73) we obtain

$$\int_{-\infty}^{\infty} dv \frac{\sum_s I_{2,1}^{(s)}(v)}{v - z_{2,1}} = \frac{1}{k^2} \int_{-\infty}^{\infty} dv \frac{\sum_s \kappa^{(s)} I_{1,0}^{(s)}}{(v - z_{2,1})^2 (v - z_{1,0})} + \mathcal{O}(\gamma^{-4}). \tag{182}$$

Now the recursion relation (68) for  $I_{1,0}^{(s)}$  is applied, and examination of the various terms shows that the dominant singularity arises from the occurrence of  $\psi^*$  in  $h_{0,0}$  so that (178) can be reduced to

$$\Gamma_{2,1} = \left( \frac{i[1 + \mathcal{O}(\gamma)]}{3\gamma k^5} \right) \int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^3 \partial_v^2 \psi^{(s)}(v)^*}{(v - z_{2,1})^2 (v - z_{1,0})} + \mathcal{O}(\gamma^{-5}). \tag{183}$$

Finally a partial fraction expansion of the remaining integral yields a  $\gamma^{-5}$  singularity

$$\int_{-\infty}^{\infty} dv \frac{\sum_s (\kappa^{(s)})^3 \partial_v^2 \psi^{(s)}(v)^*}{(v - z_{2,1})^2 (v - z_{1,0})} = \frac{1}{\gamma^5} \left[ -\left(\frac{\pi}{8}\right) k^5 \eta_k^{(p)}(v_p(0)) + \mathcal{O}(\gamma) \right], \tag{184}$$

and leads to the asymptotic form of  $p_2$  from (176)

TABLE II. Summary of the exponent  $\beta$  obtained from the amplitude equation. The cases (a), (b) and (c) are the exceptional cases discussed in Sec. V.

System	$\beta$	Comment
$c_1(0) \neq 0$	$\frac{5}{2}$	generic result for multiple mobile species
(a) $c_1(0) = 0$	2	infinitely massive ions
(b) $c_1(0) = 0$	2	$\eta_k^{(s)}(v_p) = 0$ for each ion species
(c) $c_1(0) = 0$	$\frac{9}{4} \leq \beta \leq \frac{5}{2}$	an electron-positron plasma

$$p_2 = \frac{1}{\gamma^8} \left[ \left( \frac{i\pi^2 k}{48} \right) \frac{\eta_k^{(p)}(v_p)^2}{\Lambda_k'(v_p + i0)} + \mathcal{O}(\gamma) \right] + \mathcal{O}(\gamma^{-7}). \tag{185}$$

Since  $\text{Im} \Lambda_k'(v_p + i0)$  is typically nonzero, this shows that  $\text{Re}(p_2) \sim \gamma^{-8}$  and implies that the trapping scaling (172) leaves a residual singularity at fifth order. With  $\rho(t) = \gamma^\beta r(\gamma t)$  the rescaled equation becomes

$$\frac{dr}{d\tau} = r + \gamma^{2\beta-1} \text{Re}(p_1)r^3 + \gamma^{4\beta-1} \text{Re}(p_2)r^5 + \dots \tag{186}$$

so  $\beta \geq 9/4$  is required to overcome the  $\gamma^{-8}$  divergence in  $p_2$ .

**VI. DISCUSSION**

Our analysis of an unstable electrostatic wave expresses the Vlasov distribution function  $f(x, v, t)$  in terms of the wave amplitude, i.e.,

$$f(x, v, t) = [A(t)e^{ikx}\psi(v) + \text{c.c.}] + \sum_{m=-\infty}^{\infty} e^{imkx} H_{mk}(v, A, A^*), \tag{187}$$

where  $A(t)$  evolves according to an amplitude equation,

$$\dot{A}(t) = Ap(|A|^2). \tag{188}$$

In the small growth rate limit, the eigenvalue of the unstable mode approaches the imaginary axis and, at  $\gamma=0$ , merges with a neutrally stable continuous spectrum on the axis.<sup>1,10</sup> This ‘‘interaction’’ between the unstable mode and the neutrally stable continuum leads to singular features in the asymptotic form of the amplitude equation (188).<sup>16</sup>

These singularities are physically significant since they reflect the very strong nonlinear interaction between the wave and the resonant particles at the linear phase velocity. Remarkably, the quantitative form of the singularities can be consistently interpreted as fixing overall scalings of the nonlinear solution with the linear growth rate in the asymptotic limit  $\gamma \rightarrow 0^+$ . This interpretation is achieved through a singular transformation  $|A(t)| = \gamma^\beta r(\gamma t)$  of the mode amplitude in which the exponent  $\beta$  is fixed by the criterion that the rescaled dynamics,

$$\frac{dr}{d\tau} = r(\tau) \frac{\text{Re}(p(\gamma^{2\beta} r(\tau)^2))}{\gamma}, \tag{189}$$

is free of singularities as  $\gamma \rightarrow 0^+$ . Our conclusions are summarized in Table II; with the notable exception of the electron-positron system, we can rigorously establish an exact value of  $\beta=5/2$  in the typical instability and  $\beta=2$  for certain exceptional models. In the electron-positron case, we have fixed only a range of possible  $\beta$  values from the explicit calculation of  $p(|A|^2)$  through fourth order in  $|A|$ ; the complexity of the recursion relations in this case prevents the iterative determination of the dominant singularities at each order.

The shift from  $\beta=2$  to  $\beta=5/2$  as the true asymptotic scaling due to finite mass ions at the phase velocity of the wave is our most significant conclusion regarding the physical character of the nonlinear wave. Since previous studies of the asymptotic scaling of single wave instabilities

have focused primarily on plasma waves and assumed fixed ions, this shift was not anticipated and was discovered purely as a consequence of the singularities in the amplitude equation (188). Numerical simulations of unstable ion-acoustic waves or unstable plasma waves in systems with finite mass ratios would be of considerable interest in this connection. In particular, for plasma waves one expects a crossover in the asymptotic scaling from the trapping scaling ( $\beta=2$ ) to  $\beta=5/2$  at sufficiently small growth rates. Predictions of this crossover have been made elsewhere based on the competition between the  $\gamma^{-4}$  and  $\gamma^{-3}$  singularities in the cubic coefficient.<sup>7</sup>

In this paper we have not analyzed the asymptotic form of  $f(x, v, t)$ , but it is clear from the expansions for  $H(x, v, A, A^*)$  that this is an interesting question. For example, consider the leading coefficient in  $h_0(v)$  from (77),

$$h_{0,0}(v) = -\frac{1}{k^2} \frac{\partial}{\partial v} \left[ \frac{\kappa^2 \cdot \eta_k}{(v - z_0)(v - z_0^*)} \right]; \quad (190)$$

there are poles in the complex-velocity plane at  $z_0$  and  $z_0^*$  that approach the real velocity axis at  $v_p$  in the limit  $\gamma \rightarrow 0^+$ . Thus the coefficient becomes a singular function at the phase velocity when  $\gamma=0$ ; on the other hand, for  $v \neq v_p$ , the coefficient is well-behaved at small growth rates. In the distribution function (187) this particular coefficient is multiplied by  $|A|^2$  so that after rescaling the amplitude we have a term of the form  $r^2 \gamma^{2\beta} h_{0,0}(v)$ . Now the emerging singularity in velocity space is weighted by a factor of  $\gamma^{2\beta}$ , and one would like to understand whether the resulting product is finite or singular at  $\gamma=0$ . A detailed analysis of this problem will be presented in a forthcoming paper, but the primary conclusion can be simply summarized.<sup>15</sup> Inside a neighborhood of width  $\mathcal{O}(\gamma)$ , and centered on  $v=v_p$ , the singularities in the coefficients of  $H$  are balanced by the factors of  $\gamma$  introduced by the rescaling of  $|A|$  and one can obtain finite, nonzero limits for the weighted coefficients. Outside this small neighborhood of the phase velocity, the factors of  $\gamma$  force the weighted coefficients to vanish at  $\gamma=0$ .

The connection between coefficient singularities and nonlinear scalings that is studied in this paper is relevant in other settings. As mentioned above, a key feature of our instabilities is the emergence of the critical eigenvalues from a neutrally stable continuous spectrum. It has been appreciated recently that this characteristic is shared by a diverse set of additional examples which include the onset of linear instability in fluid shear flows,<sup>17–20</sup> instabilities of solitary waves,<sup>21–23</sup> synchronization of large systems of coupled oscillators,<sup>24</sup> and bifurcations in “mean field” descriptions of the dynamics of bubble clouds in fluids.<sup>25</sup> In the phase-dynamical models of coupled oscillators, the amplitude equations of the weakly unstable modes have been analyzed in the  $\gamma \rightarrow 0^+$  limit.<sup>26,27</sup> In these studies, the coefficients can be singular or nonsingular depending on how the oscillators are coupled, but when singularities occur in the limit of small growth they can be absorbed by singular rescalings of the amplitude in a procedure analogous to the method used in this paper. The specific values of the scaling exponents are different, but yield predictions for the emergence of the nonlinear synchronized state.

It seems likely that the analysis of singular amplitude equations provides a general means to determine scaling exponents for weakly unstable systems. The exponents in turn provide a quantitative measure of the strong nonlinear interaction between the neutrally stable modes and the weakly unstable modes.

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