1. Find $\frac{dy}{dx}$.

(a) (8 points) $y = \ln\pi + \left(\sec^{-1}(\ln x)\right)^{\pi} + \pi^{\tanh x}$

$$\frac{dy}{dx} = 0 + \pi\left(\sec^{-1}(\ln x)\right)^{\pi-1} \cdot \frac{1}{|\ln x|\cdot\sqrt{(\ln x)^2+1}} \cdot \frac{1}{x} \cancel{+\pi^{\tanh x}}$$

$$+ \pi^{\tanh x} \cdot \operatorname{sech}^2 x$$

(b) (8 points) $y = \left(\coth^{-1} x\right)^{x}$ $\quad \ln y = x \ln(\coth^{-1} x)$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln(\coth^{-1}x) + x \cdot \frac{1}{\coth^{-1}x} \cdot \frac{1}{1-x^2}$$

$$\frac{dy}{dx} = \left(\ln(\coth^{-1}x) + \frac{x}{1-x^2}\cdot\frac{1}{\coth^{-1}x}\right)\cdot\left(\coth^{-1}x\right)^{x}$$

2. Evaluate the following limits.

(a) (7 points) $\lim_{x\to\infty} x\left(5^{1/x}-1\right)$ $\quad (\infty \cdot 0)$

$$= \lim_{x\to\infty} \frac{5^{\frac{1}{x}}-1}{\frac{1}{x}} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{5^{\frac{1}{x}}\cdot \ln 5 \cdot \frac{-1}{x^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} 5^{\frac{1}{x}} \cdot \ln 5 = \ln 5$$

(b) (7 points) $\lim_{x\to 0}(\cosh x)^{1/(x^2)}$ $\quad (1^{\infty})$

$$y = (\cosh x)^{1/x^2} \qquad \ln y = \frac{1}{x^2}\ln(\cosh x) = \frac{\ln(\cosh x)}{x^2}$$

$$\lim_{x\to 0} \frac{\ln(\cosh x)}{x^2} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\frac{1}{\cosh x}\cdot \sinh x}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sinh x}{x\cosh x}$$

$$\underset{L'H}{\overset{\frac{0}{0}}{=}} \frac{1}{2}\lim_{x\to 0}\frac{\cosh x}{\cosh x + x\sinh x} = \frac{1}{2}\lim_{x\to 0}\frac{1}{1+x\tanh x} = \frac{1}{2}$$

$$\lim_{x\to 0}(\cosh x)^{1/x^2} = e^{\frac{1}{2}}$$

3. Complete the following definitions. Be sure to give the domain of the function.

(a) (4 points) Let f and g be one-to-one functions. Then f and g are inverse functions provided that

$f(g(x)) = x$ for $x \in D(g)$. and $g(f(x)) = x$, for $x \in D(f)$

$R(g) = D(f)$ and $R(f) = D(g)$

(b) (3 points) $\ln x = \int_1^x \frac{dt}{t}$. $x > 0$.

(c) (3 points) For $a > 0$ and $a \neq 1$, $a^x = e^{x \ln a}$ $x \in \mathbb{R}$.

4. Evaluate the following. Your answers should be in terms of π.

a) (9 points) $\int_{1/e}^{e} \frac{dx}{x\sqrt{4-(\ln x)^2}}$ $u = \ln x$ $du = \frac{1}{x} dx$

$= \int_{-1}^{1} \frac{du}{\sqrt{4-u^2}} = \sin^{-1} \frac{u}{2} \Big|_{-1}^{1} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

b) (9 points) $\int_0^1 \frac{x+1}{x^2+3} dx$

$= \int_0^1 \frac{x}{x^2+3} dx + \int_0^1 \frac{1}{x^2+3} dx$

$u = x^2+3$

$du = 2x\,dx$

$= \frac{1}{2} \int_3^4 \frac{du}{u} + \int_0^1 \frac{dx}{3+x^2} = \frac{1}{2} \ln|u| \Big|_3^4 + \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \Big|_0^1$

$= \frac{1}{2} \ln 4 - \frac{1}{2} \ln 3 + \frac{1}{\sqrt{3}} \tan^{-1} \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \tan^{-1} 0$

$= \frac{1}{2} \ln \frac{4}{3} + \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6}$

5. Let $f(x) = (1+x^2)\tan^{-1} x$, $-\infty < x < \infty$.

(a) (5 points) Show that f has an inverse function g.

$f'(x) = 2x\tan^{-1}x + (1+x^2)\cdot\frac{1}{1+x^2} = 2x\tan^{-1}x + 1$

when $x<0$. $\tan^{-1}x<0$.
when $x>0$. $\tan^{-1}x>0$. $\Big\}$ $f'(x)>0$. increasing. One to one. inv. exists.

(b) (5 points) Find the domain of g.

$D(g) = R(f) = (f(-\infty), f(+\infty))$.

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(1+x^2)\tan^{-1}x = \infty\cdot(-\frac{\pi}{2}) = -\infty$.

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(1+x^2)\tan^{-1}x = \infty\cdot(\frac{\pi}{2}) = +\infty$.

(c) (5 points) Find $g'\left(\frac{\pi}{2}\right)$. $\therefore$ $D(R) = (-\infty, \infty)$.

$f(g(\frac{\pi}{2})) = \frac{\pi}{2}$. $f(?) = \frac{\pi}{2}$. $f(1) = 2\tan^{-1}1 = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$.

$g(\frac{\pi}{2}) = 1$. $f'(1) = 2\tan^{-1}1 + 1 = 2\cdot\frac{\pi}{4}+1 = \frac{\pi}{2}+1 = \frac{\pi+2}{2}$

$g'(\frac{\pi}{2}) = \frac{1}{f'(1)} = \frac{2}{\pi+2}$

6. (a) (4 points) Define $\cosh^{-1} x$, giving both its domain and range.

$y = \cosh^{-1}x$ iff $x = \cosh y$. $x \geq 1$. $y \geq 0$.

(b) (6 points) Use <u>your definition</u> in part (a) to derive the formula $\frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2-1}}$ (also needs to explain why you take the positive branch for $\sinh y$ or $\cosh y$).

$x = \cosh y$. $1 = \sinh y \cdot \frac{dy}{dx}$ $\frac{dy}{dx} = \frac{1}{\sinh y}$

$\cosh^2 y - \sinh^2 y = 1$. $\sinh^2 y = \cosh^2 y - 1 = x^2 - 1$

$\sinh y = \pm\sqrt{x^2-1}$. Since $y \geq 0$. $\sinh y \geq 0$.

$\sinh y = \sqrt{x^2-1}$.

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$.

7. (17 points) Suppose that the rate of growth of the earth's population is proportional to the difference between 25 billion and the existing population. If an initial census finds the population to be 5 billion and a subsequent census taken 100 years later finds the population to be 10 billion, find the population 200 years after the initial census. Express your answer as the quotient of two integers or the decimal equivalent.

Soln:

(yr) t	0	100	200
p (bil)	5	10	?

$\frac{dp}{dt} = k(25-p)$. $\frac{dp}{25-p} = k\,dt$.

$-\ln(25-p) = kt + c$. $25 - p(t) = e^{-kt} \cdot c$.

$t=0$. $p=5$. $25 - 5 = e^{0} \cdot c \Rightarrow c = 20$

$25 - p(t) = 20e^{-kt}$

$t = 100$. $p = 10$.

$25 - 10 = 20e^{-100k}$

$15 = 20e^{-100k}$. $\frac{15}{20} = e^{-100k}$.

$e^{-100k} = \frac{3}{4}$. $-100k = \ln\frac{3}{4} = -\ln\frac{4}{3}$.

$k = \frac{1}{100}\ln\frac{4}{3}$. $25 - p(t) = 20e^{-\frac{t}{100}\ln\frac{4}{3}}$

$25 - p(t) = 20\left(\frac{3}{4}\right)^{t/100}$

$t = 200$. $p = ?$

$p = 25 - 20\left(\frac{3}{4}\right)^{\frac{200}{100}} = 25 - 20\cdot\left(\frac{3}{4}\right)^2$

$= 25 - 20\cdot\frac{9}{16} = 25 - \frac{5\cdot 9}{4} = \frac{100-45}{4}$

$= \frac{55}{4} = 13.75$ (bils).

$\therefore$ The population 200 years after the initial census is 13.75 billions