$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n = a_0 \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} = a_0 e^{4x}.$$

By a computation almost identical to that in the solution of Problem 1, this series has radius of convergence $+\infty$.

C11S10.003: We use series methods to solve $2\dfrac{dy}{dx} + 3y = 0$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

so that

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Substitution in the given differential equation yields

$$\sum_{n=0}^{\infty} \left[2(n+1)a_{n+1} + 3a_n \right] x^n = 0,$$

and thus

$$a_{n+1} = -\frac{3a_n}{2(n+1)} \qquad \text{if} \quad n \geqq 0.$$

Therefore

$$a_1 = -\frac{3}{2}a_0, \qquad a_2 = -\frac{3}{2} \cdot \frac{a_1}{2} = \left(\frac{3}{2}\right)^2 \cdot \frac{a_0}{2},$$

$$a_3 = -\frac{3}{2} \cdot \frac{a_2}{3} = -\left(\frac{3}{2}\right)^3 \cdot \frac{a_0}{3!}, \qquad \ldots .$$

In general,

$$a_n = (-1)^n \left(\frac{3}{2}\right)^n \cdot \frac{a_0}{n!} \qquad \text{for} \quad n \geqq 1.$$

Therefore

$$y(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{2}\right)^n \cdot \frac{x_n}{n!} = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{3x}{2}\right)^n = a_0 e^{-3x/2}.$$

By computations quite similar to those in the solution of Problem 1, this series has radius of convergence $+\infty$.

C11S10.004: We use series methods to solve $\dfrac{dy}{dx} + 2xy = 0$. Assume that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

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$$\frac{dy}{dx} = x^2 y \qquad \text{Substitute} \quad y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$\frac{dy}{dx} = x^2 y \Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \underbrace{\sum_{n=2}^{\infty} a_{n-2} x^n}_{\text{smallest power is } x^2 \text{ term}}$$

$\therefore \; a_1 = 0, \; a_2 = 0$

for $n \geq 2$ we get

$$(n+1) a_{n+1} = a_{n-2} \Rightarrow a_{n+1} = \frac{a_{n-2}}{n+1}$$

$a_{n+1} = \frac{a_{n-2}}{n+1}$ can also be re-written as $a_{j+3} = \frac{a_j}{j+3}$ by

substituting $j = n-2$.

$\therefore$ This recurssion relation gives us ~~$a_1 = 0 \Rightarrow a_3$~~

$$a_1 = 0 \Rightarrow a_{1+3} = a_4 = 0 \Rightarrow a_7 = 0 \Rightarrow a_{10} = 0 \; \ldots$$

$$a_2 = 0 \Rightarrow a_5 = 0 \Rightarrow a_8 = 0 \Rightarrow a_{11} = 0 \; \ldots$$

$$a_3 = \frac{a_0}{3}, \quad a_6 = \frac{a_3}{6} = \frac{a_0}{6 \cdot 3}$$

$$a_9 = \frac{a_6}{9} = \frac{a_0}{9 \cdot 6 \cdot 3} = \frac{a_0}{3! \, 3^3}$$

$$a_{12} = \frac{a_9}{12} = \frac{a_0}{12 \cdot 3! \, 3^3} = \frac{a_0}{4! \, 3^4}$$

$\therefore$ in general $a_{3n} = \frac{a_0}{n! \, 3^n}$ for $n \geq 1$

Therefore

$$\cancel{y(x) = a_0\left[1 + \frac{1}{1!}\right.}$$

$$y(x) = \sum_{j=0}^{\infty} a_j x^j = \sum a_{3n} x^{3n} = \sum \frac{a_0}{n!\,3^n} x^{3n}$$

$$= a_0 \sum \left(\frac{x^3}{3}\right)^n / n! = \underline{\underline{a_0 e^{x^3/3}}}$$

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(9) $(x-1)\frac{dy}{dx} + 2y = 0$. Substitute $y = \sum_{n=0}^{\infty} a_n x^n$

$$\underbrace{(x-1)\sum_{n=1}^{\infty} n a_n x^{n-1}} + 2\sum a_n x^n = 0$$

$$\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

for $n=0$ $\quad -a_1 + 2a_0 = 0 \quad \Rightarrow \quad a_1 = 2a_0$

for $n \geq 1$ $\quad n a_n - (n+1)a_{n+1} + 2a_n = 0 \quad \Rightarrow \quad a_{n+1} = \frac{n+2}{n+1} a_n$

$\therefore \quad a_2 = \frac{3}{2}a_1 = 3a_0 \qquad a_3 = \frac{4}{3}a_2 = 4a_0$

$a_4 = \frac{5}{4}a_3 = 5a_0$

In general $\quad a_n = (n+1)a_0$

$$\therefore \quad y(x) = \sum (n+1)a_0 x^n = a_0 \sum_{n=0}^{\infty}(n+1)x^n = a_0 \sum_{n=1}^{\infty} n x^{n-1}$$

$$\cancel{\sum_{n=0}^{\infty}(n+1)x^n = 1 + \sum_{n=1}^{\infty}(n+1)x^n = 1 + \sum_{n=0}^{\infty} n x^{n-1} = 1 + \frac{d}{dx}\sum_{n=0}^{\infty} x^n}$$

$$\cancel{= 1 + \frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \text{if } |x|<1}$$

$$y(x) = a_0 \frac{d}{dx}\sum_{n=0}^{\infty} x^n = a_0 \frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \text{if } |x| < 1$$

$$y(x) = \frac{a_0}{(1-x)^2} \qquad \text{Radius of Convergence} = 1$$

Each series here has radius of convergence $+\infty$. The solution of the given differential equation can also be expressed in the form $y(x) = c_1e^{2x} + c_2e^{-2x}$.

C11S10.013: We use series methods to solve the differential equation $y'' + 9y = 0$. Assume the existence of a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \qquad \text{and}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Then substitution in the given differential equation leads—as in the solution of Problem 11—to the recursion formula $(n+2)(n+1)a_{n+2} + 9a_n = 0$, and thus

$$a_{n+2} = -\frac{9}{(n+2)(n+1)}a_n \quad \text{for} \quad n \geqq 0.$$

Hence

$$a_2 = -\frac{9}{2!}a_0, \qquad a_3 = -\frac{9}{3!}a_1,$$

$$a_4 = \frac{9^2}{4!}a_0, \qquad a_5 = \frac{9^2}{5!}a_1,$$

$$a_6 = -\frac{9^3}{6!}a_0, \qquad a_7 = -\frac{9^3}{7!}a_1,$$

and so on. Hence

$$\begin{aligned} y(x) &= a_0\left(1 - \frac{9x^2}{2!} + \frac{9^2x^4}{4!} - \frac{9^3x^6}{6!} + \cdots\right) + a_1\left(x - \frac{9x^3}{3!} + \frac{9^2x^5}{5!} - \frac{9^3x^7}{7!} + \cdots\right) \\ &= a_0\left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \cdots\right) + \frac{a_1}{3}\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots\right) \\ &= a_0\cos 3x + \frac{a_1}{3}\sin 3x = c_1\cos 3x + c_2\sin 3x. \end{aligned}$$

The radius of convergence of each series here is $R = +\infty$.

C11S10.017: Given the differential equation $x\dfrac{dy}{dx} + y = 0$, substitution of the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

as in earlier solutions in this section yields

$$\sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0;$$

$$\sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Examination of the cases $n = 0$ and $n = 1$ yields $a_0 = a_1 = 0$. If $n \geqq 2$ we see that $(n-1)a_{n-1} + a_n = 0$, and hence that $a_n = 0$ for all $n \geqq 0$. Thus the series method using the form in (1) uncovers only the trivial solution $y(x) \equiv 0$, not a general solution of the given differential equation. Part of the reason is that a general solution of the differential equation is

$$y(x) = C\exp\left(\frac{1}{x}\right).$$

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$$y'' - 2y' + y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

let $\quad y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

given $\quad y(0) = a_0 = 0$

$$\therefore \quad y(x) = \sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + \cdots$$

Since $\quad y'(x) = a_1 \cancel{x} + 2a_2 x + 3a_3 x^3 + \cdots$

and it is given that $y'(0) = 1$, we get $a_1 = 1$

$$\therefore \quad y(x) = \sum_{n=1}^{\infty} a_n x^n \quad \text{with } a_1 = 1$$

$$y'(x) = \sum_{n=2}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substituting in the differential equation yields

$$\sum (n+2)(n+1) a_{n+2} x^n - 2 \sum (n+1) a_{n+1} x^n + \sum a_n x^n = 0$$

$$\therefore \quad (n+2)(n+1) a_{n+2} - 2(n+1) a_{n+1} + a_n = 0 \qquad \text{for all } n.$$

$$a_{n+2} = \frac{2(n+1) a_{n+1} - a_n}{(n+2)(n+1)}$$

Substituting $n=0$

$$a_2 = \frac{2a_1 - a_0}{2} = \frac{2}{2} = 1 \qquad (\text{since } a_0 = 0,\ a_1 = 1)$$

$\underline{\underline{n=1}}$ gives

$$a_3 = \frac{4a_2 - a_1}{3 \cdot 2} = \frac{4-1}{3 \cdot 2} = \frac{3}{3 \cdot 2} = \frac{3}{3!}$$

$\underline{\underline{n=2}}$ gives

$$a_4 = \frac{6a_3 - a_2}{4 \cdot 3} = \frac{6 \cdot \frac{1}{2} - 1}{4 \cdot 3} = \frac{2}{4 \cdot 3} = \frac{4}{4!}$$

$\underline{\underline{n=3}}$ gives

$$a_5 = \frac{8a_4 - a_3}{5 \cdot 4} = \frac{8 \cdot \frac{1}{6} - \frac{1}{2}}{5 \cdot 4} = \frac{5/6}{5 \cdot 4} = \frac{5}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{5}{5!}$$

general form $\quad a_n = \dfrac{n}{n!}$

$$\therefore \quad y = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

$$= \sum_{n=1}^{\infty} x \cdot \frac{x^{n-1}}{(n-1)!} = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = x \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\underline{\underline{y = xe^x}}$$