Problem 1. [15 POINTS] Determine the radius and interval of convergence of the series

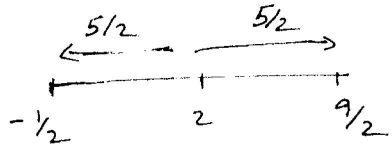
$$\sum_{n=1}^{\infty} \frac{(2x-4)^n}{n5^n}.$$

Applying ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2x-4)^{n+1}}{(n+1)5^{n+1}}}{\frac{(2x-4)^n}{n5^n}}\right| = \lim_{n\to\infty}\frac{|2x-4|}{5}\frac{n}{n+1} = \frac{|2x-4|}{5}$$

absolutely Convergent $\frac{|2x-4|}{5} < 1 \Rightarrow |2x-4| < 5$

$|x-2| < \frac{5}{2} \Rightarrow$ Radius of Convergence: $5/2$



Now for the end points

$x = -1/2$ $\quad \sum_{n=1}^{\infty}\frac{(2x-4)^n}{n5^n} = \sum\frac{(-1)^n}{n}$ Alternating hormonic Series. So Convergent

$x = \frac{1}{2}$ $\quad \sum_{n=1}^{\infty}\frac{(2x-4)^n}{n5^n} = \sum\frac{1}{n}$ divergent (hormonic Series)

$\therefore$ Interval of Convergence = $\frac{-1}{2} \leq x < 9/2$

Radius of Convergence $\underline{5/2}$ **Interval of Convergence** ______

Prob 2

1. (5pts) Give the general formula for the Taylor series of the function $f(x)$ centered at a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

2. (5pts) Find $T_3(x)$ (the third degree Taylor polynomial) for $f(x) = e^x$ centered at $a = 2$.

$$T_3(x) = e^2\left(1 + \frac{(x-a)}{1!} + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!}\right)$$

3. (5pts) Estimate the accuracy of the approximation $e^x \approx T_3(x)$ for $1.5 \leq x \leq 2.5$.

From Taylor's theorem

$$\text{Error} \leq \frac{f^{(4)}(z)}{4!}(x-a)^4 = \frac{e^z}{4!}(x-2)^4$$

$$x \in [1.5, 2.5] \qquad z \in [2, x]$$

$\therefore$ max possible error will be when $z = 2.5$ & $x = 2.5$

$$\text{Error} \leq \frac{e^{2.5}}{4!}\left(\tfrac{1}{2}\right)^4$$

Problem 3. [15 POINTS] Derive the Maclaurin series $\tan^{-1}(x)$ using the Maclaurin series formula for other known series. Compute the radius of convergence of the resulting power series. **Show all of your work.**

$$\sum t^n = \frac{1}{1-t} \quad \text{for } |t| < 1$$

So

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} \quad \text{for } |t| < 1$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\Big|_0^x \quad \text{for } |t| < 1$$

$$= \sum (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| < 1$$

$\therefore$ Radius of Convergence $= 1$

Problem 4. [20 POINTS] Solve the initial value problem

$$y'' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 3$$

using power series. **Show all of your work.**

use $y = \sum_{n=0}^{\infty} c_n x^n$ $\quad y(0) = c_0 = 0$

$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$ $\quad y'(0) = c_1 = 3$

$y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$

Substituting in the differential eqn we get the recursion relation

$$c_{n+2} = \frac{-9}{(n+2)(n+1)} c_n$$

Since $c_0 = 0 \Rightarrow c_2 = 0 \Rightarrow c_4 = 0 \Rightarrow \dots \Rightarrow c_{2n} = 0$ for all n

$c_1 = 3 \Rightarrow c_3 = \frac{-9}{3 \cdot 2} \cdot 3 = \frac{(-1)\, 3^3}{3!}$

$c_5 = \frac{-9}{5 \cdot 4} c_3 = \frac{-9}{5 \cdot 4} \frac{(-1)\, 3^3}{3!} = (-1)^2 \frac{3^5}{5!}$

$c_7 = \frac{-9}{7 \cdot 6} \cdot c_5 = (-1)^3 \frac{3^7}{7!}$ $\quad \therefore c_{2n+1} = (-1)^n \frac{3^{2n+1}}{(2n+1)!}$

$\therefore y = \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1}$ (only odd terms survive)

$$y(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = \sin(3x)$$

Its easy to see that this function does indeed satisfy the differential eqn & also the inital conditions.

Problem 5. [20 POINTS] Determine whether the following series are absolutely convergent, conditionally convergent or divergent. Justify your answers and give the name(s) of any test(s) you used to reach your conclusions.

I am just indicating the answers below. In the exam you are expected to give detailed reasoning

1. (5pts)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^{1/3}}$$

Alternating Series test $\Rightarrow$ Conditionally Convergent

2. (5pts)

$$\sum_{n=1}^{\infty} \left(\frac{n^3+7}{3n^3+6}\right)^n$$

Root test $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n^3+7}{3n^3+6}\right) = 1/3 < 1$ $\therefore$ absolutely convergent.

3. (5pts)

$$\sum_{n=1}^{\infty} \frac{2^n}{(n!)^2}$$

Ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{(n+1)^2} = 0$ $\therefore$ absolutely Convergent

4. (5pts)

$$\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2}$$

$\sum \frac{\tan^{-1}(n)}{n}$ is a positive series

Limit Comparision test. Compare with $\sum \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\tan^{-1}(n)}{n^2}}{1/n^2} = \lim_{n\to\infty} \tan^{-1}(n) = \pi/2$$

$\therefore$ absolutely Convergent.

Problem 6. [15 POINTS]

1. (4pts) Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n}$. How many terms are needed to compute the sum of the series with an error less than or equal to 10^{-6}? What is the exact value of the summation?

$n=6$ gives $\frac{(-1)^6}{10^6} = 10^{-6}$.

$\therefore \sum_{n=0}^{5} \frac{(-1)^n}{10^n}$ is sufficient to give us the desired accuracy

$\therefore$ $\underline{\underline{6 \text{ terms}}}$.

Exact Summation of $\sum_{n=0}^{\infty} \left(\frac{-1}{10}\right)^n$ is $= \frac{1}{1-\left(-\frac{1}{10}\right)} = \frac{10}{11}$ //

2. (4pts) Compute the summation

$$\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n)!} = \pi \sum (-1)^n \frac{\pi^{2n}}{(2n)!}$$

$$= \pi \cos \pi$$

$$= \underline{\underline{-\pi}}$$

3. (4pts) Consider the **sequence** $\{Q_n\}$ where $Q_n = \sum_{j=0}^{n} \frac{2^j}{j!}$. Does this **sequence** $\{Q_1, Q_2, Q_3, \ldots\}$ converge or diverge? If it is convergent, find its limit.

This Sequence is the partial Sum sequence of the Series $\sum_{j=0}^{\infty} \frac{2^j}{j!}$.

limit of the Sequence is $\sum_{j=0}^{\infty} \frac{2^j}{j!} = e^2$

4. (3pts) Given $\sum a_n = 10$, compute $\lim_{n\to\infty} e^{a_n}$.

Since $\sum a_n$ is Convergent, it must imply that

$$\lim_{n\to\infty} a_n = 0$$

$$\therefore \lim_{n\to\infty} e^{a_n} = e^0 = 1 //$$