Tests for Determining Convergence or Divergence of a Series

Definition: For $n = 1, 2, \ldots$, let $S_n = a_1 + a_2 + \ldots + a_n$.

(1) If the sequence $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ *converges* and has sum S.

(2) If the sequence $\{S_n\}$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ *diverges*.

Geometric Series Test: Consider the geometric series $a + ar + ar^2 + \ldots = \sum_{n=0}^{\infty} ar^n$, with $a \neq 0$.

(1) If $|r| < 1$, the series converges and has sum $\frac{a}{1-r}$.
(2) If $|r| \geq 1$, the series diverges.

n^{th}-term Test: If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ fails to exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(Note: If $\lim_{n\to\infty} a_n = 0$, then the n^{th}-term Test fails.)

Theorem: Suppose $\sum_{n=1}^{\infty} a_n$ converges to A and $\sum_{n=1}^{\infty} b_n$ converges to B.

(1) Then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and has sum $A + B$.

(2) Then the series $\sum_{n=1}^{\infty} ca_n$ converges for any real number c and has sum cA.

Corollary: If $\sum_{n=1}^{\infty} a_n$ diverges and c is any real number different from 0, then $\sum_{n=1}^{\infty} ca_n$ diverges.

Integral Test: Let $a_n = f(n)$, where $f(x)$ is a continuous, positive, decreasing function for $x \geq 1$.

(1) If the improper integral $\int_1^{\infty} f(x)\,dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(2) If the improper integral $\int_1^{\infty} f(x)\,dx$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

p-Series Test: Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

(1) If $p > 1$, then the series converges.
(2) If $p \leq 1$, then the series diverges.

Comparison Test: Consider the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $0 \leq a_n \leq b_n$ for all $n \geq n_o$.

(1) If the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(2) If the series $\sum_{n=1}^{\infty} a_n$ diverges, then the series $\sum_{n=1}^{\infty} b_n$ diverges.

Limit Comparison Test: Consider the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $a_n \geq 0$ and $b_n > 0$ for all $n \geq n_0$.

Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

(1) If $0 < L < \infty$, then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

(2) If $\sum_{n=1}^{\infty} b_n$ converges and $L = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(3) If $\sum_{n=1}^{\infty} b_n$ diverges and $L = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Alternating Series Test: Consider the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$.

If $a_n > a_{n+1} > 0$ for all $n \geq n_0$ and $\lim_{n\to\infty} a_n = 0$, then the series converges.

Moreover, $| S - S_n | < a_{n+1}$ for all n, where S is the sum of the series and S_n is its n^{th} partial sum.

Absolute Convergence Test: If $\sum_{n=1}^{\infty} | a_n |$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Corollary: If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} | a_n |$ diverges.

Ratio Test: Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms, and suppose that the limit

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

either exists or is infinite.

(1) If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(2) If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(3) If $\rho = 1$, then the test fails.

Root Test: Let $\sum_{n=1}^{\infty} a_n$ be a series and suppose that the limit

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

either exists or is infinite.

(1) If $\rho < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(2) If $\rho > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(3) If $\rho = 1$, then the test fails.