Math 140 - Applied work # 1
(due Monday, September 17, 2001)

1 Problem 1: Electrical resistors

Exercise 56, page 76 of the textbook.

2 Problem 2: Fluid flowing out of a draining tank

A tank like the one sketched on Fig. 1 is filled with a fluid. The level of the fluid inside the tank is denoted h. A small pipe of radius a and length L with a tap connects to the bottom of the tank. When the tap is open, the fluid flows out in the air, forming a small jet with a velocity U (averaged across the cross-section of the pipe).

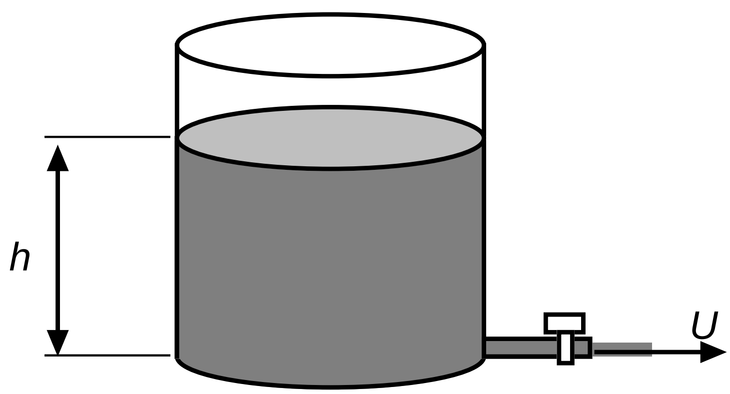

Figure 1: Sketch of the device.

The dimensions of the tank are considered very large compared to the size of the pipe, such that the velocity everywhere in the tank is neglected and the pressure distribution assumed hydrostatic. The hydrostatic pressure in the fluid at depth h is equal to:

$$p(h) = p_0 + \rho g h \tag{1}$$

where p_0 is the atmospheric pressure in the air above, ρ is the fluid density and g the acceleration of gravity.

Depending on flow conditions, the equations of fluid dynamics tell us that there may be two main regimes of flow:

- for very viscous fluids and/or very low velocities, the flow in the pipe is called *viscous flow* or *Stokes flow*;

- for low viscosity fluids (*e.g.* water) and higher velocities, the flow is *inertial* or *inviscid*.

In this problem, we will express the flow rate as a function of the depth of fluid in the tank in both regimes, and try to estimate how precisely we could control the flow rate by adjusting the height of fluid in the container.

2.1 Viscous flow

In the viscous regime, the flow in the pipe is called a *Poiseuille flow* and the average velocity in the pipe is given by:

$$U(h) = \frac{G(h)a^2}{8\eta} \tag{2}$$

where a is the radius of the pipe, η the viscosity of the fluid and $G(h)$ the pressure gradient in the pipe, *i.e.* the pressure difference per unit length of the pipe, or if you prefer, the rate of change of pressure between the ends of the pipe:

$$G(h) = \frac{\Delta p(h)}{L} \tag{3}$$

(a) Express $\Delta p(h)$, then $G(h)$, then rewrite $U(h)$ as a function of the depth h.

The flow rate $Q(h)$ of the fluid through the pipe is then simply the average velocity $U(h)$ multiplied by the cross section of the pipe.

(b) Give the expression of the flow rate $Q(h)$.

(c) If $U(h)$ is expressed in cm/s and the dimensions of the pipe in centimeters, what are the units of $Q(h)$?

We suppose the fluid is a viscous corn syrup, with a viscosity[1] $\eta = 10$ P and density $\rho = 1.2$ g/ cm^3. The pipe radius is $a = 3$ mm$= 0.3$ cm, and its length is $L = 10$ cm. The acceleration of gravity is $g = 980$ cm/s^2.

(d) What is the flow rate for $h = 1$m ($= 100$ cm) ? $h = 50$ cm ? $h = 10$ cm ?

(e) You want to maintain the flow rate around 3 cm^3/s within $\pm 10\%$ (or ± 0.3 cm^3/s). In what range should you keep the level of water in the tank to achieve this?

(f) Answer the above question if you want a precision of $\pm 1\%$

[1] the viscosity is expressed in *Poises*, which is the unit in the *c. g. s.* system where all lengths are in centimeters, masses in grams, and times in seconds.

2.2 Inviscid flow

In the inviscid case, one can use the so-called *Bernouilli's equation*, which states that the sum of the kinetic energy (per unit volume) and pressure is constant along a stream line, *i.e.* for any two points A and B on a stream line:

$$p(A) + \frac{1}{2}\rho U(A)^2 = p(B) + \frac{1}{2}\rho U(B)^2 \tag{4}$$

Consider the two points A and B on the stream line shown on Fig. 2. At A, the fluid is practically at rest, and the pressure is hydrostatic, whereas at B, the fluid exits the pipe with a velocity U at atmospheric pressure p_0.

(g) Write Bernouilli's equation for A and B and get the velocity of the fluid as a function of h.

(h) Give the expression of the flow rate $Q(h)$

(i) If the fluid is water ($\eta = 1$ cP$= 0.01$ P, $\rho = 1$ g/cm^3), calculate numerically the flow rate for $h = 100$ cm, 50 cm, 10 cm.

(j) In what range should you keep the depth h to maintain $Q(h)$ around 100 cm^3/s with $\pm 10\%$ precision ?

(k) Same question with $\pm 1\%$.

(l) Comment on the fact that it seems actually easier with this setup to control the high flow rate of a low viscosity fluid than the slow flow of a very viscous fluid. What feature of the function $Q(h)$ in both cases is responsible for that?

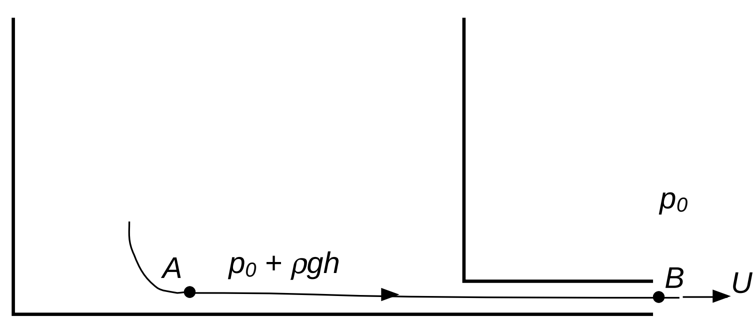

Figure 2: Stream line along which the fluid flows towards the exit. In the inviscid case, the pressure and velocity at A and B are related through equation 4.