Problem 1. [10 POINTS] Compute the fourier coefficients of the following functions:

1. $\sin(4x)$ in the interval $[-\pi, \pi]$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

Since $\sin(4x)$ is an odd function, only b_n terms survive in its fourier series.

$$b_n = \frac{1}{L} \int_{-L}^{L} \sin(4x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad L = \pi$$

$$= \begin{cases} 0 & \text{if } n \neq 4 \\ 1 & \text{if } n = 4 \end{cases}$$

$\therefore$ only b_4 is non-zero

$\therefore$ $\frac{a_0}{2} + \sum a_n \cos(nx) + \sum b_n \sin(nx) = 0 + \sum 0 + b_4 \sin(4x) = \sin(4x)$

Basically, the given function is already in the form of a fourier series.

2. $4 + x^2$ in the interval $[-3, 3]$

$f(x) = 4 + x^2$ is an even function. So only a_n terms survive

$$a_0 = \frac{1}{3} \int_{-3}^{3} (4+x^2) dx = 14$$

$$a_n = \frac{1}{3} \int_{-3}^{3} (4+x^2) \cos\left(\frac{n\pi x}{3}\right) dx = \frac{2}{3}\left[\underbrace{4\int_0^3 \cos\left(\frac{n\pi x}{3}\right) dx} + \int_0^3 x^2 \cos\left(\frac{n\pi x}{3}\right) dx\right]$$

$$= 4 \left.\frac{\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3}\right|_0^3 = 0$$

$$\int_0^3 x^2 \cos\left(\frac{n\pi x}{3}\right) dx = \left.\cancel{x^2 \frac{\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3}}\right|_0^3 - \frac{3}{n\pi} \cdot 2 \int_0^3 x \sin\left(\frac{n\pi x}{3}\right) dx = -\frac{6}{n\pi}\left[\left.\frac{-x\cos\left(\frac{n\pi x}{3}\right)}{n\pi/3}\right|_0^3 + \frac{3}{n\pi}\int_0^3 \cos\left(\frac{n\pi x}{3}\right) dx\right]$$

$$= \frac{18}{(n\pi)^2} \cdot 3 \cdot \cos(n\pi) = \frac{54}{(n\pi)^2}(-1)^n$$

$$\therefore \quad a_n = \frac{2}{3} \cdot \frac{54}{(n\pi)^2}(-1)^n = \frac{36}{(n\pi)^2}(-1)^n$$

Problem 2. [15 POINTS] Find the inverse of the given linear transformation

$$\begin{aligned}\tilde{x} &= 4x - 2y + 2z\\ \tilde{y} &= -2x - 4y + 4z\\ \tilde{z} &= -4x + 2y + 8z\end{aligned}$$

Show all of your work.

$$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & -4 & 4 \\ -4 & 2 & 8 \end{pmatrix} \qquad \tilde{X} = AX$$

Work through the elementary transformations to get A^{-1}.

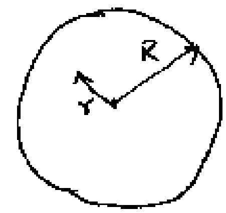
$$A^{-1} = \begin{pmatrix} 1/5 & -1/10 & 0 \\ 0 & -1/5 & 1/10 \\ 1/10 & 0 & 1/10 \end{pmatrix}$$

∴ Inverse transformation is

$$X = A^{-1}\tilde{X}$$

Problem 3. [15 POINTS] The acceleration due to gravity inside the earth is proportional to the distance from the center of the earth. An object is dropped from the surface of the earth into a hole extending through the earth's center to the other side. Calculate the time taken by the object to reach the other side of the earth's surface. **Show all of your work.**

$r'' \propto r$



(1) — $r'' = -\omega^2 r$ (the proportionality const is negative)

Initial Conditions

$t=0 \Rightarrow r=R$
$t=0 \Rightarrow r'=0$ } dropped from the surface of earth.

Solution to eqn (1) $r = A\cos(\omega t) + B\sin(\omega t)$

$t=0,\ r=B$ gives $R=B$

$t=0,\ r'=0$ (zero initial velocity) gives

$0 = \omega A\cos(\omega t) - B\omega\sin(\omega t)\big|_{t=0} \Rightarrow A=0$

$\therefore\ r = R\cos(\omega t)$

So the object undergoes simple harmonic motion.

$\therefore$ time period of oscillation $T = 2\pi/\omega$

Time to reach other side $= T/2 = \pi/\omega$

Problem 4. [20 POINTS] Use the method of variation of parameters to find the general solution of the non-homogeneous system of differential equations

$$X' = \begin{pmatrix} 2 & 0 \\ 5 & 2 \end{pmatrix} X + \begin{pmatrix} 2 \\ 10t \end{pmatrix}$$

Show all of your work.

$(A-\lambda I) = \begin{pmatrix} 2-\lambda & 0 \\ 5 & 2-\lambda \end{pmatrix}$, $|A-\lambda I| = (2-\lambda)^2 = 0 \Rightarrow \lambda = 2$ double root

Eigenvector: $(A-\lambda I)E = 0 \Rightarrow x_1 = 0 \Rightarrow E_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ only one genuine eigenvector.

Determine pseudoeigenvector E_2 by $(A-\lambda I)E_2 = E_1 \Rightarrow \begin{matrix} 0x_1 + 0x_2 = 0 \\ 5x_1 + 0x_2 = 1 \end{matrix} \Rightarrow x_1 = 1/5$

$E_2 = \begin{pmatrix} 1/5 \\ 0 \end{pmatrix}$

$\varphi_1 = e^{\lambda t}E_1 = e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\varphi_2 = e^{\lambda t}(E_1 t + E_2) = e^{2t}\begin{pmatrix} 1/5 \\ t \end{pmatrix}$

$\Omega = e^{2t}\begin{pmatrix} 0 & 1/5 \\ 1 & t \end{pmatrix}$ $X_h = \Omega C$

Particular Solution $\Psi_p = \Omega u$ where $u' = \Omega^{-1} h$ $h = \begin{pmatrix} 2 \\ 10t \end{pmatrix}$

$|\Omega| = -\frac{1}{5}e^{4t}$ $\Omega^{-1} = \frac{1}{|\Omega|}e^{2t}\begin{pmatrix} t & -1/5 \\ 1 & 0 \end{pmatrix} = e^{-2t}\begin{pmatrix} -5t & 1 \\ -5 & 0 \end{pmatrix}$

$u' = \Omega^{-1}h = e^{-2t}\begin{pmatrix} 0 \\ -10 \end{pmatrix} \Rightarrow u = \begin{pmatrix} 0 \\ 5e^{-2t} \end{pmatrix}$

$\Psi_p = \Omega u = \begin{pmatrix} 1 \\ 5t \end{pmatrix}$

$X = X_h + \Psi_p = e^{2t}\left[c_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1/5 \\ t \end{pmatrix}\right] + \begin{pmatrix} 1 \\ 5t \end{pmatrix}$ //

Problem 5. [20 POINTS] Solve the system

$$X' = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} X$$

1. Find three linearly independent solutions to the ODE system
2. Prove that your solutions in previous part are indeed independent.
3. Find the general solution of the ODE system.

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)^2 = 0 \Rightarrow \lambda_1 = 3, \quad \lambda_2 = \lambda_3 = 2$$

Now we Compute Eigenvectors

$\lambda_3 = \lambda_2 = 2 \quad (A-\lambda I)E_2 = 0 \Rightarrow$
$$\begin{aligned} (3-2)x_1 + x_2 + 0 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \Rightarrow x_1 + x_2 = 0 \Rightarrow \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ \& } \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{\text{2 independent eigenvector}}$$

$\lambda_1 = 3 \quad (A-\lambda I)E_1 = 0 \Rightarrow$
$$\begin{aligned} x_1 &= 0 \\ -x_2 &= 0 \\ -x_3 &= 0 \end{aligned} \Rightarrow E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ ~~general~~ Solutions are $\varphi_1 = E_1 e^{3t} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t}$

$\varphi_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} \qquad \varphi_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t}$

They are indeed independent because

$$|E_1\, E_2\, E_3| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0$$

(If they were not independent, then the determinant would have been zero)

Problem 6. [20 POINTS] Find a real-valued fundamental matrix for the system of ordinary differential equations:

$$X' = \begin{pmatrix} 3 & -2 \\ 5 & -3 \end{pmatrix} X$$

Show all of your work.

$$(A-\lambda I) = \begin{pmatrix} 3-\lambda & -2 \\ 5 & -3-\lambda \end{pmatrix}$$

$$|A-\lambda I| = (3-\lambda)(-3-\lambda)+10 = -(9-\lambda^2)+10 = \lambda^2+1 = 0 \qquad \lambda = \pm i = \alpha + i\beta$$

$$\alpha = 0, \quad \beta = 1$$

$$(A-\lambda_1 I)E_1 = \begin{matrix} (3-i)x_1 - 2x_2 = 0 \\ 5x_1 - (3+i)x_2 = 0 \end{matrix} \Rightarrow x_2 = \frac{(3-i)}{2} x_1$$

$$E_1 = \begin{pmatrix} 2 \\ 3-i \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad V = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\varphi_1(t) = e^{\alpha t}\left[U\cos(\beta t) - V\sin(\beta t)\right] = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\cos(t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin(t) = \begin{pmatrix} 2\cos(t) \\ 3\cos(t) + 5\sin(t) \end{pmatrix}$$

$$\varphi_2(t) = e^{\alpha t}\left[U\sin(\beta t) + V\cos(\beta t)\right] = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\sin(t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos(t) = \begin{pmatrix} 2\sin(t) \\ 3\sin(t) - \cos(t) \end{pmatrix}$$

$\frac{?}{?}$ cos t ∗ $\frac{3}{5}$ sin t − 2 sin t

$$\Omega = \begin{pmatrix} 2\cos t & 2\sin t \\ 3\cos t + \sin t & 3\sin t - \cos t \end{pmatrix}$$

$$\begin{pmatrix} -\sin t & -\frac{3}{5}\sin(t) - \frac{1}{5}\cos(t) \\ -\frac{3}{2}\sin t + \frac{1}{2}\cos t & -\sin t \end{pmatrix}$$